DOCUMENT RESUME

ED 135 629 SE 021 999

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TITLE Elementary Functions, Student's Text, Unit 21.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 61
NOTE 398p.; For related documents, see SE 021 987-022 002 and ED 130 870-877; Contains occasional light type

EDRS PRICE MF-$0.83 HC-$20.75 Plus Postage.
DESCRIPTORS *Curriculum; Elementary Secondary Education; Instruction; *Instructional Materials; Mathematics Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS *Functions (Mathematics); *School Mathematics Study Group

ABSTRACT

Unit 21 in the SMSG secondary school mathematics series is a student text covering the following topics in elementary functions: functions, polynomial functions, tangents to graphs of polynomial functions, exponential and logarithmic functions, and circular functions. Appendices discuss set notation, mathematical induction, significance of polynomials, area under a polynomial graph, slopes of area functions, the law of growth, approximation and computation of e raised to the x power, an approximation for ln x, measurement of triangles, trigonometric identities and equations, and calculation of sin x and cos x. (DT)

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SE 021 999

SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

School Mathematics Study Group

Elementary Functions

Unit 21

Elementary Functions

Student's Text

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

Below are listed the names of all those who participated in any of the writing sessions at which the following SMSG texts were prepared: *First Course in Algebra*, *Geometry*, *Intermediate Mathematics*, *Elementary Functions*, and *Introduction to Matrix Algebra*.

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PREFACE

This experimental text is intended for use in a one semester 12th grade course. The appendices provide material for supplementary study by able students. Alternatively, by including some or all of the material in the appendices it is possible to use this text for a longer course.

The central theme is a study of functions. The first chapter, Functions, gives a background for the study of Polynomial Functions (Chapters 2 and 3), Exponential and Logarithmic Functions (Chapter 4), and Circular (or Trigonometric) Functions (Chapter 5).

The introductory chapter uses a small amount of set notation which is explained in an appendix. A function is defined in terms of the concept of mapping and, to a certain extent, in terms of a computing machine. After a treatment of constant and linear functions, composition and inversion of functions are carefully discussed.

Chapter 2 covers standard material on the algebra of polynomials, but the treatment is a more modern and rigorous one than is found in conventional texts.

Chapter 3 is concerned with the use of the tangent line as an approximation to the graph of a polynomial function near a given point. The equation of the tangent at this point is obtained by algebraic procedures which are both simple and logically precise. These procedures also give a method for determining the shape of the graph nearby. The student is thus able to locate critical points and to solve interesting maximum and minimum problems. Our treatment will furnish him with a good background for a later course in calculus. An appendix gives an introduction to the problem of finding areas under graphs.

In Chapter 4 the characteristic features of exponential growth are brought out in an intuitive way, with applications to problems of current interest. The method for finding tangents explained in Chapter 3 is used to obtain the slopes of exponential graphs. Logarithmic functions are defined as the inverses of exponential functions.

Chapter 5 deals with the theory of the circular functions in the spirit of the previous chapters, emphasizing the study of periodic motion and the analytic properties of the trigonometric functions. An appendix includes supplementary material for students who have had no previous course in trigonometry.

This text is an attempt to implement the recommendations of the Commission on Mathematics of the C.E.E.B. The authors are conscious of its shortcomings and welcome criticism from those who may use it.

Chapter 1

FUNCTIONS

1-1. Functions

One of the most useful and universal concepts in mathematics is that of a function, and this book, as its title indicates, will be devoted to the study of functions, with particular attention to a few special functions that are of fundamental importance.

We frequently hear people say, "One function of the Police Department is to prevent crime," or "Several of my friends attended a social function last night," or "My automobile failed to function when I tried to use it." In mathematics we use the word "function" somewhat differently than we do in ordinary conversation; as you have probably learned in your previous study, we use it to denote a certain kind of association or correspondence between the members of two sets.

We find examples of such associations on every side. For instance, we note such an association between the number of feet a moving object travels and the difference in clock readings at two separate points in its journey; between the length of a steel beam and its temperature; between the price of eggs and the cost of making a cake. Additional examples of such associations occur in geometry, where, for instance, we have the area or the circumference of a circle associated with the length of its radius.

In all of these examples, regardless of their nature, there seems to be the natural idea of a direct connection of the elements of one set to those of another; the set of distances to the set of times, the set of lengths to the set of thermometer readings, etc. It seems natural, therefore, to abstract from these various cases this idea of association or correspondence and examine it more closely.

Let us start with some very simple examples. Suppose we take the numbers 1, 2, 3, and 4, and with each of them associate the number twice as large: with 1 we associate 2, with 2 we associate 4, with 3 we associate 6, and with 4 we associate 8. An association such as this is called a function, and the set $\{1, 2, 3, 4\}$ with which we started is called the domain of the function (for a summary of set notation see Appendix to Chapter 1). We can represent this association more briefly if we use arrows instead of words: $1 \longrightarrow 2$, $2 \longrightarrow 4$, $3 \longrightarrow 6$, $4 \longrightarrow 8$. There are, of course, many other functions with the same domain; for example, $1 \longrightarrow 2$, $2 \longrightarrow 1$, $3 \longrightarrow 2$, $4 \longrightarrow 5$.

It happens that these two examples deal with numbers, but there are many functions which do not. A map, for instance, associates each point on some bit of terrain with a point on a piece of paper; in this case, the domain of the function is a geographical region. We can, indeed, generalize this last example, and think of any function as a mapping; thus, our first two examples map numbers into numbers, and our third maps points into points.

What are the essential features of each of these examples? First, we are given a set, the domain. Second, we are given a rule of some kind which associates an object of some sort with each element of the domain, and, third, we are given some idea of where to find this associated object. Thus, in the first example above, we know that if we start with a set of real numbers, and double each, the place to look for the result is in the set of all real numbers. To take still another example, if the domain of a function is the set of all real numbers, and the rule is "take the square root," then the set in which we must look for the result is the set of complex numbers. We summarize this discussion in the following definition:

Definition 1-1. If with each element of a set A there is associated in some way exactly one element of a set B, then this association is called a function from A to B.

It is common practice to represent a function by the letter "f" (other letters such as "g" and "h" will also be used). If x is an element of the domain of a function f, then the object which f associates with x is denoted $f(x)$ (read "the value of f at x" or simply "f at x" or "f of x"); $f(x)$ is called the image of x. Using the arrow notation of our examples, we can represent this symbolically by

$$f: x \longrightarrow f(x)$$

(read "f takes x into f(x)"). This notation tells us nothing about the function f or the element x; it is merely a restatement of what "f(x)" means.

The set A mentioned in Definition 1-1 is, as has been stated, the domain of the function. The set of all objects onto which the function maps the element of A is called the range of the function; in set notation (see Appendix),

$$\text{Range of } f = \{f(x): x \in A\}.$$

The range may be the entire set B mentioned in the definition, or may be only a part thereof, but in either case it is included in B.

It is often helpful to illustrate a function as a mapping, showing the elements of the domain and the range as points and the function as a set of arrows from the points that represent elements of the domain to the points that represent elements of the range, as in Figure 1-1a. Note that, as a consequence of Definition 1-1, to each element of the domain there corresponds one and only one element of the range. If this condition is not met, as in Figure 1-1b, then the mapping pictured is not a function. In terms of the pictures, a mapping is not a function if two arrows start from one point; whether two arrows go to the same point, as in Figure 1-1a, is immaterial in the definition. This requirement, that each element of the domain be mapped onto one and only one element of the range, may seem arbitrary, but it turns out, in practice, to be

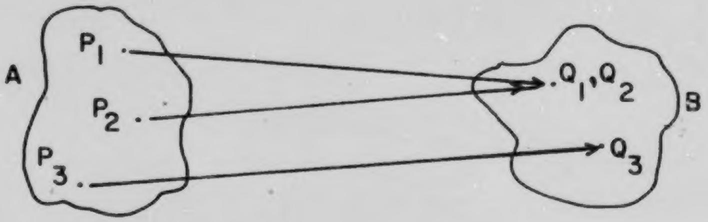


Figure 1-1a. A function as a mapping.

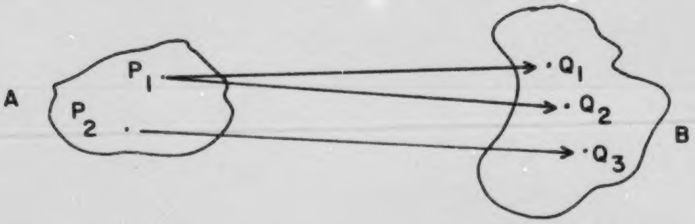


Figure 1-1b. This mapping is not a function.

extremely convenient.

In this book, we are primarily concerned with functions whose domain and range are sets of real numbers, and we shall therefore assume, unless we make explicit exception, that all of our functions are of this nature. It is therefore convenient to represent the domain by a set of points on a number line and the range as a set of points on another number line, as in Figure 1-1c.

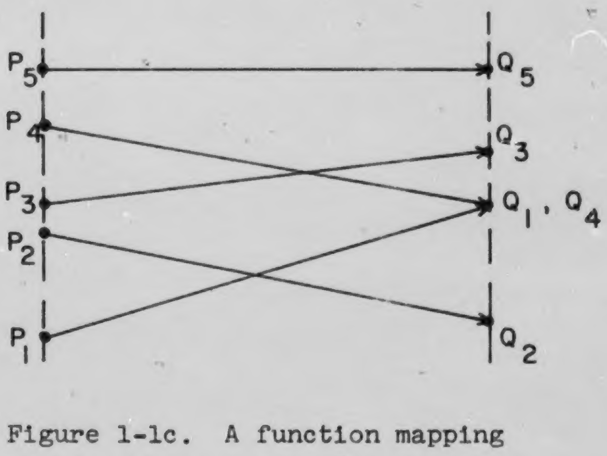


Figure 1-1c. A function mapping real numbers into real numbers.

More specifically, consider the function f, discussed earlier, which takes each element of the set {1, 2, 3, 4} into the number twice as great. The range of this function is {2, 4, 6, 8} and f maps its domain onto its range as shown in Figure 1-1d. We note that, in this case, the image of the element x of the domain of f is the element 2x; hence we may write, in this instance, $f(x) = 2x$, and f is completely specified by the notation

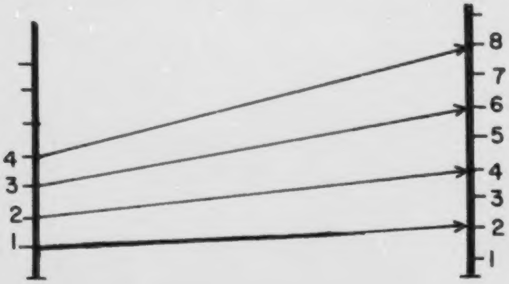


Figure 1-1d. $f: x \longrightarrow 2x, \quad x = 1, 2, 3, 4.$

$$f: x \longrightarrow 2x, \quad x = 1, 2, 3, 4.$$

In this case, the way in which f maps its domain onto its range is completely specified by the formula $f(x) = 2x$. Most of the functions which we shall consider can similarly be described by appropriate formulas. If, for example, f is the function that takes each number into its square, then it takes 2 into 4 (that is, $f(2) = 4$), it takes -3 into 9 (that is, $f(-3) = 9$), and in general, it takes any real number x into x^2. Hence, for this function, $f(x) = x^2$, we may write $f: x \longrightarrow x^2$. The formula $f(x) = x^2$ defines this function f, and to find the image of any element of the domain, we can merely substitute in this formula; thus, if $a - 3$ is a real number, then $f(a-3) = (a-3)^2 = a^2 - 6a + 9$. Similarly, if we know that a function f has $f(x) = 2x - 3$ for all

$x \in R$ (we use R to represent the set of real numbers) then we can represent f in our mapping notation as $f\colon x \longrightarrow 2x - 3$, and to find the image of any real number we need only substitute it for x in the expression $2x - 3$; thus $f(5) = 2(5) - 3 = 7$, $f(\sqrt{2}) = 2\sqrt{2} - 3$, and if $k + 2$ is a real number, then

$$f(k + 2) = 2(k + 2) - 3 = 2k + 1.$$

Strictly speaking, a function is not completely described unless its domain is specified. In dealing with a formula, however, it is a common and convenient practice to assume, if no other information is given, that the domain includes all real numbers that yield real numbers when substituted in the formula. For example, if nothing further is said, in the function $f\colon x \longrightarrow 1/x$, the domain is assumed to be the set of all real numbers except 0; this exception is made because $1/0$ is not a real number. Similarly, if f is a function such that $f(x) = \sqrt{1 - x^2}$, we assume, in the absence of any other information, that the domain is $\{x\colon -1 \le x \le 1\}$, that is, the set of all real numbers from -1 to $+1$ inclusive, since only these real numbers will give us real square roots in the expression for $f(x)$. When a function is used to describe a physical situation, the domain is understood to include only those numbers that are physically realistic. Thus, if we are describing the volume of a balloon in terms of the length of its radius, $f\colon r \longrightarrow V$, the domain would include only positive numbers.

A humorist once defined mathematics as "a set of statements about the twenty-fourth letter of the alphabet." We may not agree about just how funny this statement is, but we must agree that it contains an element of truth: we do make "x" work very hard. It is important to recognize that this arises out of custom, not necessity, and that any other letter or symbol would do just as well. The notations $f\colon x \longrightarrow x^2$, $f\colon h \longrightarrow h^2$, $f\colon t \longrightarrow t^2$, and even $f\colon \# \longrightarrow \#^2$ all describe exactly the same function, subject to our agreement that x, h, t, or $\#$ stands for any real number.

Another way of looking at a function, which may help you to understand this section, is to think of it as a machine that processes elements of its domain to produce elements of its range. The machine has an input and an output; if an element x of its domain is fed on a tape into the machine, the element f(x) of the range will appear as the output, as indicated in Figure 1-1e.

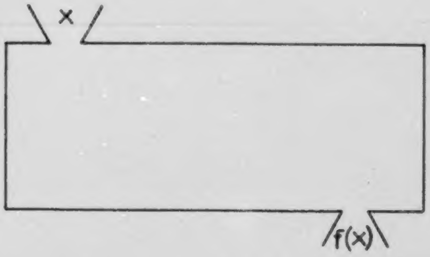


Figure 1-1e.
A representation of a function as a machine.

A machine can only be set to perform a predetermined task. It cannot exercise judgment, make decisions, or modify its instructions. A function machine f must be set so that any particular input x always results in the same output f(x); if the element x is not in the domain of f, the machine will jam or refuse to perform. Some machines -- notably computing machines -- actually do work in almost exactly this way.

Exercises 1-1

1. Which of the following do not describe functions, when $x, y \in R$?
   - a) $f: x \longrightarrow 3x - 4$
   - b) $f: x \longrightarrow x^3$
   - c) $f: x \longrightarrow -x$
   - d) $f: x \longrightarrow y = x^2$
   - e) $f: x \longrightarrow$ all $y < x$
   - f) $f: x \longrightarrow 5x$
   - g) $f: x \longrightarrow 16 - x^2$
2. Depict the mapping of a few elements of the domain into elements of the range for each of the Exercises 1(a), (c), and (d) above, as was done in Figure 1-1d.

3. Specify the domain and range of the following functions, where $x, f(x) \in R$.
   - a) $f: x \longrightarrow x$
   - b) $f: x \longrightarrow x^2$
   - c) $f: x \longrightarrow \sqrt{x}$
   - d) $f: x \longrightarrow \frac{x}{x - 1}$
   - e) $f: x \longrightarrow \frac{3}{x^2 - 4}$
4. If $f: x \longrightarrow 2x + 1$, find
   - a) $f(0)$
   - b) $f(-1)$
   - c) $f(100)$
   - d) $f(\frac{3}{2})$
5. Given the function $f: x \longrightarrow x^2 - 2x + 3$, find
   - a) $f(0)$
   - b) $f(-1)$
   - c) $f(a)$
   - d) $f(x - 1)$
6. If $f(x) = \sqrt{x^2 - 16}$, find
   - a) $f(4)$
   - b) $f(-5)$
   - c) $f(5)$
   - d) $f(a)$
   - e) $f(a - 1)$
   - f) $f(\pi)$
7. If $f: x \longrightarrow \frac{4}{3} x^3 - 12x^2 + \frac{98}{3} x - 20$ has the domain $\{1, 2, 3, 4\}$, a) find the image of f, and b) depict f as in Figure 1-1d.
8. If $x \in R$, given the functions

   $$f: x \longrightarrow x$$

   and

   $$g: x \longrightarrow \frac{x^2}{x}$$

   are f and g the same function? Why or why not?
9. What number or numbers have the image 16 under the following functions?
   - a) $f: x \longrightarrow x^2$
   - b) $f: x \longrightarrow 2x$
   - c) $f: x \longrightarrow \sqrt{x^2 + 112}$

1-2. The Graph of a Function.

A *graph* is a set of points. If the set consists of all points whose coordinates (x, y) satisfy an equation in x and y, then the set is said to be the graph of that equation. If there is a function f such that, for each point (x, y) of the graph, and for no other points, we have $y = f(x)$, then we say that the graph is the graph of the function f. The graph is perhaps the most intuitively illuminating representation of a function; it conveys at a glance much important information about the function. The function $x \longrightarrow x^2$, (when there is no danger of confusion, we sometimes omit the name of a function, as "f" in $f: x \longrightarrow x^2$) has the parabolic graph shown in Figure 1-2a. We can look at the parabola and get a clear intuitive idea of what the function is doing to the elements of its domain. We can, moreover, usually infer from the graph any limitations on the domain and the range. Thus, it is clear from Figure 1-2a, that the range of the function there graphed includes only non-negative numbers, and in the function $f: x \longrightarrow \sqrt{25 - x^2}$ graphed in Figure 1-2b, the domain $\{x: -5 \leq x \leq 5\}$ and range $\{y: 0 \leq y \leq 5\}$ are easily determined, as shown by the heavy segments on the x-axis and y-axis, respectively.

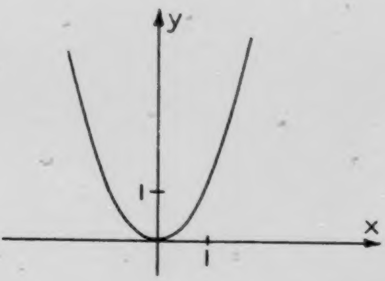


Figure 1-2a.
Graph of the function $f: x \longrightarrow x^2$

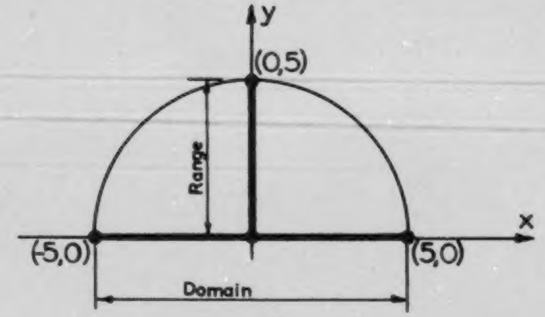


Figure 1-2b.
Graph of the function $f: x \longrightarrow \sqrt{25 - x^2}$

Another illustration: the function

$$f: x \longrightarrow x/2, \quad 2 < x \leq 6$$

has domain $A = \{x : 2 < x \leq 6\}$ and range $B = \{f(x) : 1 < f(x) \leq 3\}$. In this case we have used open dots at 2 on the x axis and at 1 on the y-axis to indicate that these numbers are not elements of the domain and range respectively. See Figure 1-2c.

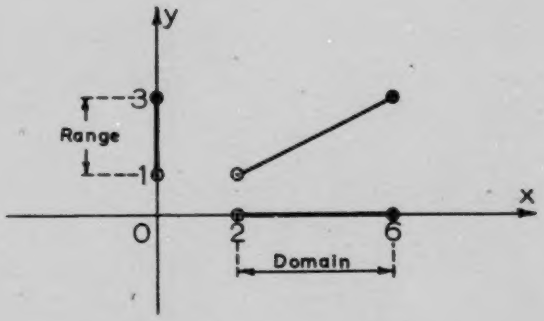


Figure 1-2c.
Graph of the function $f: x \longrightarrow x/2, \; 2 < x \leq 6$.

As might be expected, not every possible graph is the graph of a function. In particular, Definition 1-1 requires that a function map each element of its domain onto only one element of its range. In the language of graphs, this says that only one value of y can correspond to any value of x. If, for example, we look at the graph of the equation $x^2 + y^2 = 25$, shown in Figure 1-2d, we can

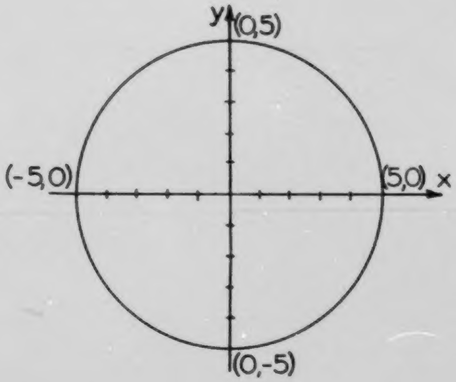


Figure 1-2d.
Graph of the set $S = \{(x,y) : x^2 + y^2 = 25\}$.

see that there are many instances in which one value of x is associated with two values of y, contrary to the definition of function. To give a specific example, if $x = 3$, we have both $y = 4$ and $y = -4$; each of the points $(3, 4)$ and $(3, -4)$ is on the graph. Hence this is not the graph of a function. We can, however, break it into two pieces, the graph of $y = \sqrt{25 - x^2}$ and the graph of $y = -\sqrt{25 - x^2}$ (this makes the points $(-5, 0)$ and $(5, 0)$ do double duty), each of which is the graph of a function. See Figures 1-2e and 1-2f.

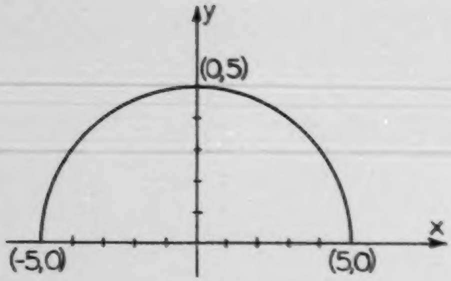


Figure 1-2e.
Graph of $y = \sqrt{25 - x^2}$.

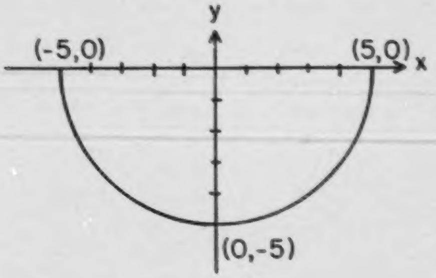
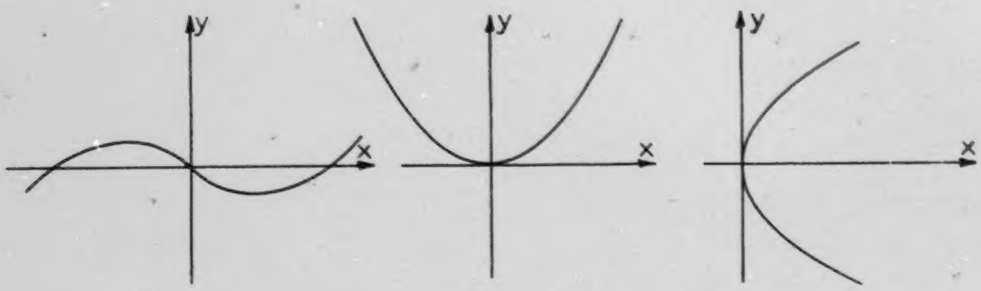


Figure 1-2f.
Graph of $y = -\sqrt{25 - x^2}$.

If, in the xy-plane, we imagine all possible lines which are parallel to the y-axis, and if any of these lines cuts the graph in more than one point, then the graph defines a relation that is not a function. Thus, in Figure 1-2g, (a) depicts a function, (b) depicts a function, but (c) does not depict a function.



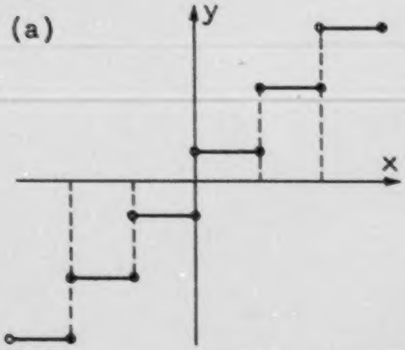
(a) (b) (c)

Figure 1-2g. Function or not?

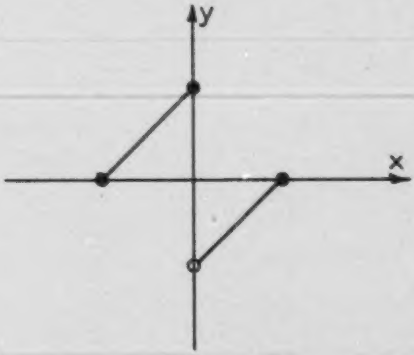
Exercises 1-2

1. Which of the following graphs could represent functions?

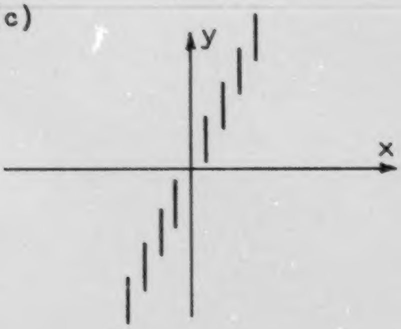
(a)



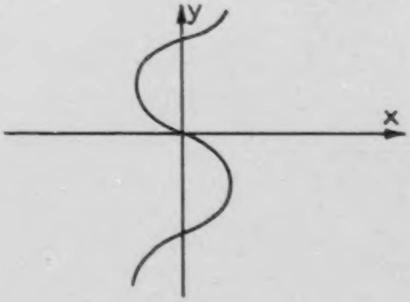
(b)



(c)



(d)



2. Suppose that in (a) above, $f: x \longrightarrow f(x)$ is the function whose graph is depicted. Sketch
   a) $g: x \longrightarrow -f(x)$ b) $g: x \longrightarrow f(-x)$
3. Graph the following functions.
   a) $f: x \longrightarrow 2x$
   b) $f: x \longrightarrow \frac{1}{x}$
   c) $f: x \longrightarrow y = 4 - x$ and x and y are positive integers.
   d) $f: x \longrightarrow -\sqrt{4 - x^2}$

4. Graph the following functions and indicate the domain and range of each by heavy lines on the x-axis and y-axis respectively.
   a) $f: x \longrightarrow y = x$ and $2 < y < 3$
   b) $f: x \longrightarrow \sqrt{9 - x^2}$
   c) $f: x \longrightarrow \sqrt{x}$ and $x < 4$

1-3. Constant Functions and Linear Functions

We have introduced the general idea of function, which is a particular kind of an association of elements of one set with elements of another. We have also interpreted this idea graphically for functions which map real numbers into real numbers. In Sections 1-1 and 1-2 our attention was concentrated on general ideas, and examples were introduced only for the purposes of illustration. In the present section we reverse this emphasis and study some particular functions that are important in their own right. We begin with the simplest of these, namely the constant functions and the linear functions.

Let us think of a man walking north along a long straight road at the uniform rate of 2 miles per hour. At some particular time, say time $t = 0$, this man passed the milepost located one mile north of Baseline Road. An hour before this, which we shall call time $t = -1$, he passed the milepost located one mile south of Baseline Road. An hour after time $t = 0$, at time $t = 1$, he passed the milepost located three miles north of Baseline Road. In order to form a convenient mathematical picture of the man's progress, let us consider miles north of Baseline Road as positive and miles south as negative. Thus the man passed milepost -1 at time $t = -1$, milepost 1 at time $t = 0$, and milepost 3 at time $t = 1$. Using an ordinary set of coordinate axes let us plot his position, as indicated by the mileposts, versus time in hours. This gives us the graph shown in Figure 1-3a.

In t hours the man travels $2t$ miles. Since he is already at milepost 1 at time $t = 0$, he must be at milepost $2t + 1$ at time t. This pairing of numbers is an example of a linear function.

Now let us plot the man's speed versus time. For all values

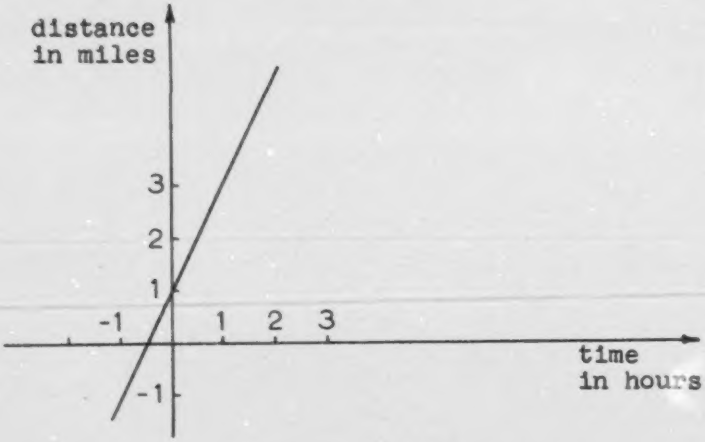


Figure 1-3a.
Graph of the function $f: t \longrightarrow d = 2t + 1$

of t during the time he is walking his speed is 2 miles per hour. We have graphed this information in Figure 1-3b. When

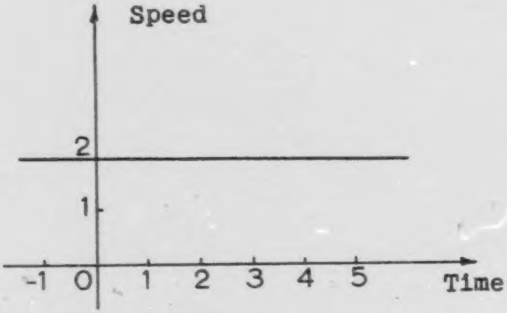


Figure 1-3b.
Graph of the function $g: t \longrightarrow s = 2$.

$t = -1$ his speed is 2, when $t = 0$ his speed is 2, etc.; with each number t we associate the number 2. This mapping, in which the range contains only the one number 2, is an example of a constant function.

Definition 1-2. If with each real number x we associate one fixed number c, then the resultant mapping,

$$f: x \longrightarrow c,$$

is called a constant function.

The discussion of constant functions can be disposed of in a few lines. The function we just mentioned, for example, is the constant function $g: t \longrightarrow 2$. The graph of any constant function is a line parallel to the horizontal x-axis. Constant functions are very simple, but they occur over and over again in mathematics and science and are really quite important. A well-known example from physics is the magnitude of the attraction of gravity, which is usually taken to be constant over the surface of the earth -- though, in this age, we must recognize the fact that the attraction of gravity varies greatly throughout space.

The functions we examine next also occur over and over again in mathematics and science and are considerably more interesting than the constant functions. These are the linear functions. Since you have worked with these functions before, we can begin at once with a formal definition.

Definition 1-3. A function f defined on the set of all real numbers is called a linear function if there exist real numbers m and b, with $m \neq 0$, such that

$$f(x) = mx + b.$$

Example 1. The function $f: x \longrightarrow 2x + 1$ is a linear function. Here $f(0) = 1$, $f(1) = 3$, $f(-1) = -1$. This function was described earlier in this section in terms of t, with $f(t) = 2t + 1$. Its graph can be found in Figure 1-3a.

We note that the graph in Figure 1-3a appears to be a straight line. As a matter of fact, the graphs of all linear functions are straight lines (that is why we call them "linear" functions); you may be familiar with a proof of this theorem from an earlier study of graphs. In any case, we here assume it.

An important property of any straight line segment is its slope, defined as follows.

Definition 1-4. The slope of the line segment from the point $P(x_1, y_1)$ to the point $Q(x_2, y_2)$ is the number

$$\frac{y_2 - y_1}{x_2 - x_1},$$

provided $x_1 \neq x_2$. If $x_1 = x_2$, the slope is not defined.

Note that, by Definition 1-4, the slope of the line segment from the point $Q(x_2, y_2)$ to the point $P(x_1, y_1)$ is

$$\frac{y_1 - y_2}{x_1 - x_2} .$$

But

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} ,$$

so it is immaterial which of the two points P or Q we take first. Accordingly, we can speak of $(y_2 - y_1)/(x_2 - x_1)$ as the slope of the segment joining the two points, without specifying which comes first.

What about the geometric meaning of the slope of a segment? Suppose, for the sake of definiteness, we consider the segment joining $P(1, 2)$ and $Q(3, 8)$. By our definition, the slope of this segment is 3, since $(8 - 2)/(3 - 1) = 3$ (or $(2 - 8)/(1 - 3) = 3$). Note that this is the vertical distance from P to Q divided by the horizontal distance from P to Q, or, in more vivid language, the rise divided by the run.

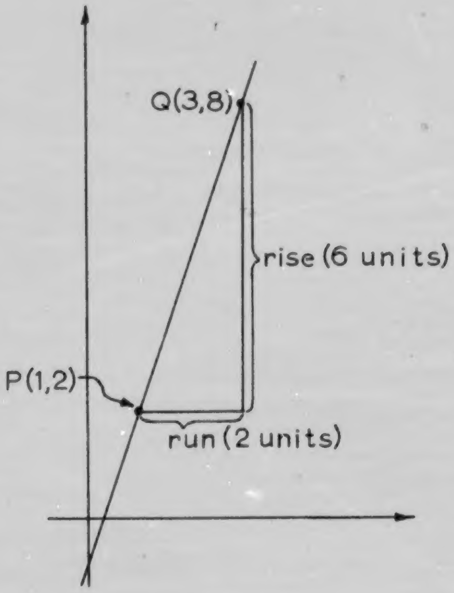


Figure 1-3c.

Let us think of the segment PQ as running from left to right, so that the run is positive. If the segment rises, then the "rise" is positive and the slope, or ratio of rise to run, is positive; if, on the other hand, the segment falls, then the "rise" is negative, and the slope is therefore negative. The steeper the segment, the larger is the absolute value of its slope, and conversely; thus we can use the slope as a numerical measure of the "steepness" of a segment.

We have stated that slope is not defined if $x_1 = x_2$; in this case, the segment lies on a line parallel to the y-axis. It is important to distinguish this situation from the case $y_1 = y_2$ (and $x_1 \neq x_2$), in which a slope is defined and in fact has value 0; the segment is then on a line parallel to the x-axis.

If a line is the graph of a linear function $f\colon x \longrightarrow mx + b$, then for any x_1 and x_2, $x_1 \neq x_2$, the slope of the segment joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is, by definition,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = m;$$

in other words, the slope m is independent of the choice of x_1 and x_2, and is therefore the same for every segment of the line. Hence we may consider the slope to be a property of the line as a whole, rather than of a particular segment. We shall also simplify our language a little and speak of the slope of the graph of a function as, simply, the slope of the function. We see, moreover, that we can read the slope of a linear function directly from the expression which defines the function: the slope of $f\colon x \longrightarrow mx + b$ is simply m, the coefficient of x. Thus, the slope of the linear function $f\colon x \longrightarrow 2x + 1$ is 2, the coefficient of x, and, similarly, the slope of $g\colon x \longrightarrow -5x$ is -5.

Since the slope of a linear function $f\colon x \longrightarrow mx + b$ is the number $m \neq 0$, it follows that the graph of a linear function is not parallel to the x-axis. Conversely, it can be proved that any line not parallel to either axis is the graph of some linear function. We assume that this, also, is known to you from previous work, and the proof is therefore omitted.

If the graphs of the functions $f_1\colon x \longrightarrow m_1x + b_1$ and

$f_2: x \longrightarrow m_2x + b_2$ meet, there must be a value of x which satisfies the equation $f_1(x) = f_2(x)$, that is,

$$m_1x + b_1 = m_2x + b_2,$$

or

$$(m_1 - m_2)x = b_2 - b_1.$$

If $m_1 \neq m_2$, then the value $x = (b_2 - b_1)/(m_1 - m_2)$ satisfies this equation, and the lines do indeed meet. If $m_1 = m_2$ and $b_1 = b_2$, the functions f_1 and f_2 are the same, and there is only one line. If $m_1 = m_2$ and $b_1 \neq b_2$, the equation has no solution, and the lines do not meet. We conclude that lines with the same slope are parallel, and that two lines parallel to each other but not to the y-axis have equal slopes.

Note that lines having zero slope, that is, lines parallel to the x-axis, are graphs of constant functions. On the other hand, lines for which no slope is defined, that is, lines parallel to the y-axis, cannot be graphs of any functions because, with one value of x, the graph associates more than one value -- in fact, all real values.

Example 2. Find the linear function g whose graph passes through the point with coordinates $(-2, 1)$ and is parallel to the graph of the function $f: x \longrightarrow 3x - 5$.

Solution. The graph of f is a line with slope 3. Hence the slope of g is the number 3, so that $g(x) = 3x + b$, for some as yet unknown b. Since $g(-2) = 1$, this implies that $1 = 3(-2) + b$, $b = 7$, and thus $g(x) = 3x + 7$ for all $x \in R$.

Exercises 1-3

1. Find the slope of the function f if, for all real numbers x,
   a) $f(x) = 3x - 7$
   b) $f(x) = 6 - 2x$
   c) $2f(x) = 3 - x$
   d) $3f(x) = 4x - 2$
2. Find a linear function f whose slope is -2 and such that
   a) $f(1) = 4$
   c) $f(3) = 1$
   b) $f(0) = -7$
   d) $f(8) = -3$

3. Find the slope of the linear function f if $f(1) = -3$ and
   a) $f(0) = 4$
   b) $f(2) = 3$
   c) $f(5) = 5$
   d) $f(6) = -13$
4. Find a function whose graph is the line joining the points
   a) $P(1, 1)$, $Q(2, 4)$
   b) $P(-7, 4)$, $Q(-5, 0)$
   c) $P(1, 3)$, $Q(1, 8)$
   d) $P(1, 4)$, $Q(-2, 4)$
5. Given $f: x \longrightarrow -3x + 4$, find a function whose graph is parallel to the graph of f and passes through the point
   a) $P(1, 4)$
   b) $P(-2, 3)$
   c) $P(1, 5)$
   d) $P(-3, -4)$
6. If f is a constant function find $f(3)$ if
   a) $f(1) = 5$
   b) $f(8) = -3$
   c) $f(0) = 4$
7. Do the points $P(1, 3)$, $Q(3, -1)$, and $S(7, -9)$ all lie on a single line? Prove your assertion.
8. The graph of a linear function f passes through the points $P(100, 25)$ and $Q(101, 39)$. Find
   a) $f(100.1)$
   b) $f(100.3)$
   c) $f(101.7)$
   d) $f(99.7)$
9. The graph of a linear function f passes through the points $P(53, 25)$ and $Q(54, -19)$. Find
   a) $f(53.3)$
   b) $f(53.8)$
   c) $f(54.4)$
   d) $f(52.6)$
10. Find a linear function with graph parallel to the line with equation $x - 3y + 4 = 0$ and passing through the point of intersection of the lines with equations $2x + 7y + 1 = 0$ and $x - 2y + 8 = 0$.
11. Given the points $A(1, 2)$, $B(5, 3)$, $C(7, 0)$, and $D(3, -1)$, prove that $ABCD$ is a parallelogram.
12. Find the coordinates of the vertex C of the parallelogram $ABCD$ if AC is a diagonal and the other vertices are the points:
    a) $A(1, -1)$, $B(3, 4)$, $D(2, 3)$;
    b) $A(0, 5)$, $B(1, -7)$, $D(4, 1)$
13. If t is a real number, show that the point $P(t + 1, 2t + 1)$ is on the graph of $f: x \longrightarrow 2x - 1$.

14. If you graph the set of all ordered pairs of the form $(t - 1, 3t + 1)$ for $t \in R$ you will obtain the graph of a linear function f. Find $f(0)$ and $f(8)$.
15. If you graph the set of all ordered pairs of the form $(t - 1, t^2 + 1)$ for $t \in R$, you will obtain the graph of a function f. Find $f(0)$ and $f(8)$.
16. If the slope of a linear function f is negative, prove that $f(x_1) > f(x_2)$ for $x_1 < x_2$.

1-4. The Absolute-value Function

A function of importance in many branches of mathematics is the absolute-value function, $f: x \longrightarrow |x|$ for all $x \in R$. The absolute value of a number describes the size, or magnitude, of the number, without regard to its sign; thus, for example $|2| = |-2| = 2$ (read "$|2|$" as "the absolute value of 2"). A common definition of $|x|$ is

Definition 1-5.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

A consequence of this definition is that no number has a negative absolute value ($-x$ is positive when x is negative); in fact, the range of the absolute-value function is the entire set of non-negative real numbers.

A very convenient alternative definition of absolute value is the following:

Definition 1-6. $|x| = \sqrt{x^2}$

Since we shall make use of this definition in what follows, it is important that you understand it, and you must therefore be quite sure of the meaning of the square-root symbol, $\sqrt{}$. This never indicates a negative square root. Thus, for example, $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3; $\sqrt{x}$ is never negative. It is true that every positive number has two real square roots, one of them positive and the other negative, but the symbol $\sqrt{}$ has been assigned the job of representing the positive root only, and if we wish to represent the negative root, we must use a minus sign before the radical. Thus, for example, the number 5 has two square roots, $\sqrt{5}$ and $-\sqrt{5}$.

The graph of the absolute-value function is shown in Figure 1-4a.

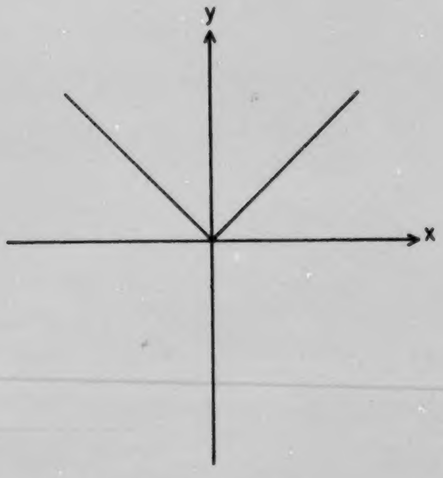


Figure 1-4a.
Graph of the function $f: x \longrightarrow |x|$

You should be able to see, from the first definition of this function given above, that this graph consists of the origin, the part of the line $y = x$ that lies in Quadrant I, and the part of the line $y = -x$ that lies in Quadrant II.

There are two important theorems about absolute values.

Theorem 1-1. For any two real numbers a and b, $|ab| = |a| \cdot |b|$.

Proof: $|a| \cdot |b| = \sqrt{a^2}\ \sqrt{b^2} = \sqrt{a^2b^2} = \sqrt{(ab)^2} = |ab|$.

Theorem 1-2. For any two real numbers a and b, $|a + b| \leq |a| + |b|$.

Proof: By Definition 1-6, Theorem 1-2 is equivalent to

$$\sqrt{(a+b)^2} \leq \sqrt{a^2} + \sqrt{b^2}, \qquad (1)$$

which is equivalent to

$$a^2 + 2ab + b^2 \leq a^2 + 2\sqrt{a^2}\sqrt{b^2} + b^2,$$

and hence to $2ab \leq 2\sqrt{a^2}\sqrt{b^2}$

or $ab \leq \sqrt{a^2}\sqrt{b^2}$. (2)

Now equation (2) is easy to prove. If a and b have opposite signs, then $ab < 0$ and (2) holds with the $<$ sign. Otherwise, we have $ab = \sqrt{a^2}\ \sqrt{b^2}$.

Hence in any case $ab \leq \sqrt{a^2}\ \sqrt{b^2}$,

and therefore (1) holds. q.e.d.

Thus, for example, $|(-2)(3)| = |-6| = 6 = 2 \cdot 3 = |-2| \cdot |3|$, $|(-2) + (3)| = 1 < 5 = 2 + 3 = |-2| + |3|$, and $|(-2) + (-3)| = 5 = 2 + 3 = |-2| + |-3|$.

Exercises 1-4

1. a) For what $x \in R$ is it true that $\sqrt{x^2} = x$?
   b) For what $x \in R$ is it true that $\sqrt{x^2} = -x$?
2. a) For what $x \in R$ is it true that $|x - 1| = x - 1$?
   b) For what $x \in R$ is it true that $|x - 1| = -x + 1$?
   c) Sketch a graph of $f: x \longrightarrow |x - 1|$.
   d) Sketch a graph of $f: x \longrightarrow |x| - 1$.
3. Solve:
   a) $|x| = 14$
   b) $|x + 2| = 7$
   c) $|x - 3| = -1$
4. For what values of x is it true that
   a) $|x - 2| < 1$
   b) $|x - 5| > 2$
   c) $|x + 4| < 0.2$
   d) $|2x - 3| < 0.04$
   e) $|4x + 5| < 0.12$
5. Show that $x^2 \geq x \cdot |x|$ for all $x \in R$.
6. Show that $|a - b| \leq |a| + |b|$.
7. Show that $\frac{1}{2}(a + b + |a - b|)$ is equal to the greater of a and b. Can you write a similar expression for the lesser of a and b?
8. Sketch: $y = |x| + |x - 2|$. (Hint: you must consider, sepa-

rately, the three possibilities $x < 0$, $0 \leq x < 2$, and $x \geq 2$.)

9. If $0 < x < 1$, we can multiply both sides of the inequality $x < 1$ by the positive number x to obtain $x^2 < x$, and we can similarly show that $x^3 < x^2$, $x^4 < x^3$, and so on. Use this result to show that if $|x| < 1$, then $|x^2 + 2x| < 3|x|$.
10. Show that, if $0 < x < k$, then $x^2 < kx$. Hence show that, if $|x| < 0.1$, then $|x^2 - 3x| < 3.1|x|$.
11. For what values of x is it true that $|x^2 + 2x| < 2.001|x|$?

1-5. Composition of Functions

Our consideration of functions, to this point, has been concerned with individual functions, with their domains and ranges, and with their graphs. We now consider certain things that can be done with two or more functions somewhat as, when we start school, we first learn numbers and then learn how to combine them in various ways. There is, as a matter of fact, a whole algebra of functions, just as there is an algebra of numbers. Functions can be added, subtracted, multiplied, and divided. The sum of two functions f and g, for example, is defined to be the function

$$f+g: x \longrightarrow f(x) + g(x)$$

which has for domain the intersection of the domains of f and g; there are similar definitions, which you can probably supply yourself, for the difference, product, and quotient of two functions. Because, for example, the number $(f+g)(x)$ can be found by adding the numbers $f(x)$ and $g(x)$, it follows that this part of the algebra of functions is so much like the familiar algebra of numbers that it would not pay us to examine it carefully. There is, however, one important operation in this algebra of functions that has no counterpart in the algebra of numbers: the operation of composition.

The basic idea of composition of two functions is that of a kind of "chain reaction" in which the functions occur one after the other. Thus, an automobile driver knows that the amount he depresses the accelerator pedal controls the amount of gasoline fed to the cylinders and this in turn affects the speed of the car. Again, the momentum of a rocket sled when it is near the end of its runway depends on the velocity of the sled, and this in turn depends on the thrust of the propelling rockets.

Let us look at a specific illustration. Suppose that f is the function $x \longrightarrow 3x - 1$ (this might be a time-velocity function) and suppose that g is the function $x \longrightarrow 2x^2$ (this might be a velocity-energy function). Let us follow what happens when we "apply" these two functions in succession--first f, then g--to a particular number, say the number 4. In brief, let us first calculate $f(4)$ and then calculate $g(f(4))$. (Read this "g of f of 4".)

First calculate $f(4)$. Since f is the function $x \longrightarrow 3x - 1$, $f(4) = 3 \cdot 4 - 1 = 11$. Then calculate $g(f(4))$, or $g(11)$. Since g is the function $x \longrightarrow 2x^2$, $g(11) = 2 \cdot 11^2 = 242$. Thus $g(f(4)) = g(11) = 242$. In general, $g(f(x))$ is the result we obtain when we first "apply" f to an element x and then "apply" g to the result. The function $x \longrightarrow g(f(x))$ is then called a composite of f and g, and denoted gf.

We say a composite rather than the composite because the order in which these functions occur is important. To see that this is the case, start with the number 4 again, but this time find $g(4)$ first, then $f(g(4))$. The results are as follows:

$g(4) = 2 \cdot 4^2 = 32$ and $f(g(4)) = f(32) = 3 \cdot 32 - 1 = 95$. Clearly $g(f(4))$, which is 242, is not the same as $f(g(4))$, which is 95.

Warning. When we write "gf" we mean that f is to be applied before g and then g is applied to $f(x)$. Since "f" is written after "g" is written, this can easily lead to confusion. You can avoid the confusion by thinking of the equation $(gf)(x) = g(f(x))$.

It may be helpful to diagram the above process as follows: If gf is the function $x \longrightarrow g(f(x))$ and fg is the function $x \longrightarrow f(g(x))$ we have

$$\begin{tikzcd} 4 \arrow[r, "f"] \arrow[rr, bend left, "gf"] & 11 \arrow[r, "g"] & 242 \end{tikzcd} \qquad \begin{tikzcd} 4 \arrow[r, "g"] \arrow[rr, bend left, "fg"] & 32 \arrow[r, "f"] & 95 \end{tikzcd}$$

Note particularly that fg is not the product of f and g mentioned earlier in this section. When we want to talk about this product, $f \cdot g$, we shall always use the dot as shown. Incidentally, for the above example, we have $(f \cdot g)(4) = f(4) \cdot g(4) = 11 \cdot 32 = 352 = 32 \cdot 11 = g(4) \cdot f(4) = (g \cdot f)(4)$.

To generalize this illustration, let us use x instead of 4 and find algebraic expressions for $(gf)(x)$ and $(fg)(x)$. We do this as follows:

$$(gf)(x) = g(f(x)) = g(3x - 1) = 2\cdot(3x - 1)^2$$

and $$(fg)(x) = f(g(x)) = f(2x^2) = 3(2x^2) - 1 = 6x^2 - 1.$$

Again, note that $(gf)(x)$ and $(fg)(x)$ are not the same so the function gf is not the same as the function fg. In symbols, $gf \neq fg$. If, now, we substitute 4 for x we obtain

$$(gf)(4) = 2(3 \cdot 4 - 1)^2 = 242$$

and $$(fg)(4) = 6 \cdot 4^2 - 1 = 95$$

These results agree with the ones we obtained above.

We are now ready to define the general process that we have been illustrating.

Definition 1-7. Given two functions, f and g, the function $x \longrightarrow g(f(x))$ is called a composite of f and g and denoted gf. The domain of gf is the set of all elements x in the domain of f for which $f(x)$ is in the domain of g. The operation of forming a composite of two functions is called composition.

Example 1. Given that $f\colon x \longrightarrow 3x - 2$ and $g\colon x \longrightarrow x^5$ for all $x \in R$, find

a) $(gf)(x)$　　c) $f(g(x) + 3)$

b) $(ff)(x)$　　d) $f(g(x) - f(x))$

Solution:

a) $(gf)(x) = g(f(x)) = g(3x - 2) = (3x - 2)^5$

b) $(ff)(x) = f(f(x)) = f(3x - 2) = 3(3x - 2) - 2 = 9x - 8$

c) $f(g(x) + 3) = f(x^5 + 3) = 3(x^5 + 3) - 2 = 3x^5 + 7$

d) $f(g(x) - f(x)) = f(x^5 - 3x + 2) = 3(x^5 - 3x + 2) - 2 = 3x^5 - 9x + 4$

If we think of a function as a machine with an input and an output, as suggested in Section 1-1, we see that two such machines can be arranged in tandem, so that the output of the first machine feeds into the input of the second. This results in a "composite" process that is analogous to the operation of composition. It is illustrated in Figure 1-5b. In this figure the machine for f and the machine for g have been housed in one cabinet. This compound machine is the machine for gf.

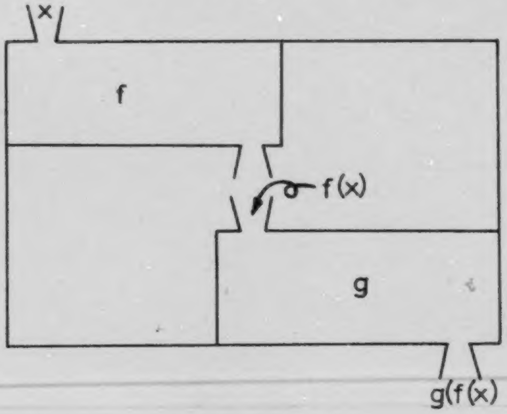


Figure 1-5b.
Schematic representation of the composition of functions.

Note that the machine for gf will jam if either of two things happens:

a) It will jam if a number not in the domain of f is fed into the machine.

b) It will jam if the output f(x) of f is not in the domain of g.

Thus, once again we see that the domain of gf is the set of all elements x in the domain of f for which f(x) is in the domain of g.

We have noted that the operation of composition is not commutative; that is, it is not always true that $fg = gf$. On the other hand, it is true that this operation is associative: for any three functions f, g, and h, it is always true that $(fg)h = f(gh)$. We shall not prove this theorem; we shall however, illustrate its operation by an example.

Example 2. Given $f: x \longrightarrow x^2 + x + 1$, $g: x \longrightarrow x + 2$, and $h: x \longrightarrow -2x - 3$, find

a) fg	c) (fg)h

b) gh	d) f(gh)

Solution:

a) $(fg)(x) = (x + 2)^2 + (x + 2) + 1 = x^2 + 5x + 7$, so
   $fg: x \longrightarrow x^2 + 5x + 7$

b) $(gh)(x) = (-2x - 3) + 2 = -2x - 1$, so $gh: x \longrightarrow -2x - 1$

c) $(fg)h: x \longrightarrow (-2x - 3)^2 + 5(-2x - 3) + 7$

d) $f(gh): x \longrightarrow (-2x - 1)^2 + (-2x - 1) + 1$

It is not altogether obvious from these expressions that (fg)h and f(gh) are the same function. But if you will simplify the expressions you will see that they are indeed the same.

Exercises 1-5

1. Given that $f: x \longrightarrow x^2 - 1$ and $g: x \longrightarrow x + 2$ for all $x \in R$, find
   - a) $(fg)(-2)$
   - b) $(gf)(0)$
   - c) $(gg)(1)$
   - d) $(ffg)(1)$
   - e) $(fg)(x)$
   - f) $(gf)(x)$
   - g) $\dfrac{(fg)(x) - (fg)(1)}{x - 1}$ if $x \neq 1$

2. Let it be given that $f: x \longrightarrow ax + b$ and $g: x \longrightarrow cx + d$ for all $x \in R$.
   - a) Find $(fg)(x)$.
   - b) Find $(gf)(x)$.
   - c) Compare the slopes of fg and gf with the slopes of f and g.
   - d) Formulate a theorem concerning the slope of a composite of two linear functions.

3. Suppose that $f: x \longrightarrow 1/x$ for all real numbers x different from zero.
   - a) Find $(ff)(1)$, $(ff)(-3)$, and $(ff)(8)$.
   - b) Describe ff completely.

4. Let it be given that $j: x \longrightarrow x$ and $f: x \longrightarrow x + 2$ for all $x \in R$.
   - a) Find fj and jf. [First find $(fj)(x)$ for all $x \in R$.]

b) Find a function g such that $fg = j$. [That is, find g such that $(fg)(x) = j(x)$ for all $x \in R$.]

c) Find a function h such that $hf = j$. Compare your result with that of (b).

5. a) If $f: x \longrightarrow x^2$ and $g: x \longrightarrow x^3$, find expressions for $(fg)(x)$ and $(gf)(x)$.

b) If $f: x \longrightarrow x^m$ and $g: x \longrightarrow x^n$, find expressions for $(fg)(x)$ and $(gf)(x)$.

6. a) If $f: x \longrightarrow x^2$ and $g: x \longrightarrow x^3$, find an expression for $(f \cdot g)(x)$, where $f \cdot g$ is the product of f and g; that is, $(f \cdot g)(x) = f(x) \cdot g(x)$. Compare with Exercise 5(a).

b) If $f: x \longrightarrow x^m$ and $g: x \longrightarrow x^n$ for all $x \in R$ (where m and n are positive integers), find an expression for $(f \cdot g)(x)$. Compare with Exercise 5(b).

7. Suppose that $f: x \longrightarrow x + 2$, $g: x \longrightarrow x - 3$, and $h: x \longrightarrow x^2$ for all $x \in R$. Find expressions for

a) $(f \cdot g)(x)$ d) $(gh)(x)$
b) $[(f \cdot g)h](x)$ e) $[(fh) \cdot (gh)](x)$
c) $(fh)(x)$

8. In Exercise 7, compare your results for (b) and (e). They should be the same. Do you think this result is true for any three functions f, g, and h, that map real numbers into real numbers?

9. Would you say that $f(g \cdot h) = (fg) \cdot (fh)$ for any three functions f, g, and h, that map real numbers into real numbers

10. State which of the following will hold for all functions f, g, and h, that map real numbers into real numbers:

$$(f+g)h = fh + gh$$
$$f(g+h) = fg + fh$$

11. Prove that the set of all linear functions is associative under composition; that is, for any three linear functions f, g, and h,

$$f(gh) = (fg)h$$

1-6. Inversion

Quite frequently in science and in everyday life we encounter quantities that bear a kind of reciprocal relationship to each other. With each value of the temperature of the air in an automobile tire, for example, there is associated one and only one value of the pressure of the air against the walls of the tire. Conversely, with each value of the pressure there is associated one and only one value of the temperature. Two more examples, numerical ones, will be found below.

Suppose that f is the function $x \longrightarrow x + 3$ and g is the function $x \longrightarrow x - 3$. Then the effect of f is to increase each number by 3, and the effect of g is to decrease each number by 3. Hence f and g are reciprocally related in the sense that each undoes the effect of the other. If we add 3 to a number and then subtract 3 from the result we get back to the original number. In symbols

$$(gf)(x) = g(f(x)) = g(x + 3) = (x + 3) - 3 = x.$$

Similarly,

$$(fg)(x) = f(g(x)) = f(x - 3) = (x - 3) + 3 = x.$$

As a slightly more complicated example we may take

$$f: x \longrightarrow 2x - 3 \text{ and } g: x \longrightarrow \frac{x + 3}{2}.$$

Here f says "Take a number, double it, and then subtract 3." To reverse this, we must add three and then divide by 2. This is the effect of the function g. In symbols,

$$(gf)(x) = g(f(x)) = g(2x - 3) = \frac{(2x - 3) + 3}{2} = x.$$

Similarly,

$$(fg)(x) = f(g(x)) = f(\frac{x + 3}{2}) = 2\frac{x + 3}{2} - 3 = x.$$

In terms of our representation of a function as a machine, the g machine in each of these examples is equivalent to the f machine running backwards; each machine then undoes what the other does, and if we hook up the two machines in tandem, every element that gets through both will come out just the same as it originally went in.

We now generalize these two examples in the following definition of inverse functions.

Definition 1-8. If f and g are functions so related that $(fg)(x) = x$ for every element x in the domain of g and $(gf)(y) = y$ for every element y in the domain of f, then f and g are said to be inverses of each other. In this case both f and g are said to have an inverse, and each is said to be an inverse of the other.

As a further example of the concept of inverse functions let us examine the functions $f: x \longrightarrow x^3$ and $g: x \longrightarrow \sqrt[3]{x}$. In this case

$$(fg)(x) = f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

and

$$(gf)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

for all $x \in R$.

If a function f takes x into y, that is, if $y = f(x)$, then an inverse g of f must take y right back into x, that is, $x = g(y)$. If we make a picture of a function as a mapping, with an arrow extending from each element of the domain to its image, as in Figure 1-6a, then to draw a picture of the inverse function we need merely reverse the arrows, as in Figure 1-6b.

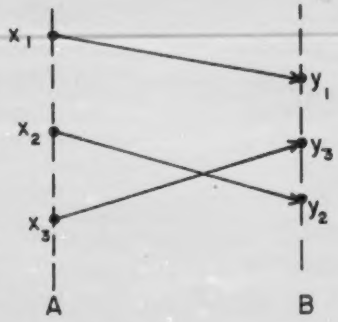


Figure 1-6a. A function.

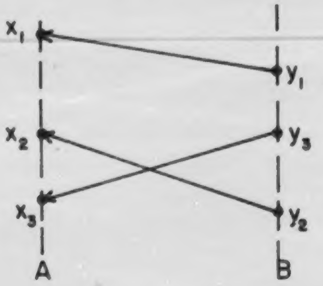


Figure 1-6b. Its inverse.

We can take any mapping, reverse the arrows in this way, and obtain another mapping. The important question for us, at this point, is this: If the original mapping is a function, will the reverse mapping necessarily be a function also? In other words, given a function, does there exist another function that precisely reverses the effect of the given function? We shall see that this is not always the case.

The definition of a function (Definition 1-1) requires that to each element of the domain there corresponds *exactly one* element of the range; it is perfectly all right for several elements of the domain to be mapped onto the same element of the range (the constant function, for example, maps *all* of its domain onto one element), but if even one element of the domain is mapped onto more than one element of the range, then the mapping just isn't a function. In terms of a picture of a function as a mapping (such as Figures 1-1a and 1-1c), this means that no two arrows may start from the same point, though any number of them may end at the same point. But if two or more arrows go to one point, as in Figure 1-6c, and if we then reverse the arrows, as in Figure 1-6d, we will have two or more arrows *starting* from that point (as in Figure 1-1b), and the resulting mapping is not a function. Since the word "inverse" is used to describe only a mapping which is a *function*, we can conclude that not every function has an inverse.

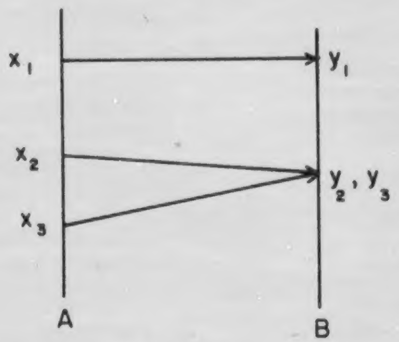


Figure 1-6c.

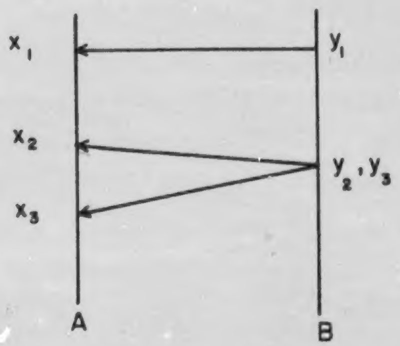


Figure 1-6d.

A specific example is furnished by the constant function f: $x \longrightarrow 3$; since $f(0) = 3$ and $f(1) = 3$, an inverse of f would have to map 3 onto both 0 and 1. By definition, no function can do this.

The preceding argument shows us just what kinds of functions do have inverses. By comparing the situation in Figures 1-6a and 1-6b with the situation in Figures 1-6c and 1-6d, we can see that a function has an inverse if and only if no two arrows go to the same point. In more precise language, a function f has an inverse if and only if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. A function of this sort is often called a "one-to-one" function. A formal proof of this Theorem will be found in Chapter 4.

Exercises 1-6

1. Find an inverse of each of the following functions:
   a) $x \longrightarrow x - 7$
   b) $x \longrightarrow 5x + 9$
   c) $x \longrightarrow 1/x$
2. Solve each of the following equations for x in terms of y and compare your answers with those of Exercise 1:
   a) $y = x - 7$
   b) $y = 5x + 9$
   c) $y = 1/x$
3. Justify the following in terms of composite functions and inverse functions: Ask someone to choose a number, but not to tell you what it is. "Ask the person who has chosen the number to perform in succession the following operations. (i) To multiply the number by 5. (ii) To add 6 to the product. (iii) To multiply the sum by 4. (iv) To add 9 to the product. (v) To multiply the sum by 5. Ask to be told the result of the last operation. If from this product 165 is subtracted, and then the difference is divided by 100, the quotient will be the number thought of originally."
   (W. W. Rouse Ball).

1-7. Summary of Chapter 1.

This chapter deals with functions in general and with the constant and linear functions in particular.

A function is an association between the objects of one set, called the domain, and those of another set, called the range, such that exactly one element of the range is associated with each element of the domain. A function can be represented as a mapping from its domain to its range.

The graph of a function is often an aid to understanding the function. A graph is the graph of a function if and only if no line parallel to the y-axis meets it in more than one point.

A constant function is an association of the form $f: x \longrightarrow c$, for some fixed real number c, with the set of all real numbers as its domain. The graph of a constant function is a straight line parallel to the x-axis.

A linear function is an association of the form $f: x \longrightarrow mx + b$, $m \neq 0$. The domain and the range of a linear function are each the set of all real numbers. The graph of a linear function is a straight line not parallel to either axis, and, conversely, any such line is the graph of some linear function.

The slope of the line through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

if $x_1 \neq x_2$. If $x_1 = x_2$, no slope is defined, and the line is parallel to the y-axis. Lines with the same slope are parallel, and parallel lines which have slopes have equal slopes. The slope of the graph of the linear function $f: x \longrightarrow mx + b$ is the coefficient of x, namely the constant m.

The absolute-value function is conveniently defined as $f: x \longrightarrow \sqrt{x^2}$. The domain of this function is the set of all real numbers and the range is the set of all non-negative real numbers.

If f and g are functions, then the composite function fg is $fg: x \longrightarrow f(g(x))$, with domain all x in the domain of g such that $g(x)$ is in the domain of f.

Given a function f, if there exists a function g such that (gf)(y) = y for all y in the domain of f and (fg)(x) = x for all x in the domain of g, then g is an inverse of f. Not all functions have inverses; those that do are called one-to-one functions.

Miscellaneous Exercises

1. Which table defines a function $f: x \longrightarrow y$?

a)

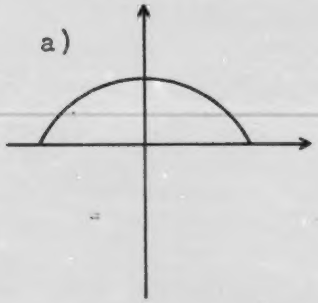
x	1	2	3
y	1	2	2

b)

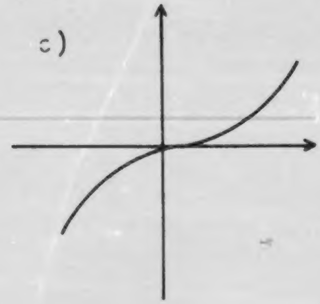
x	1	2	2
y	1	2	3

2. Which graphs represent functions? Which of these functions have inverses?

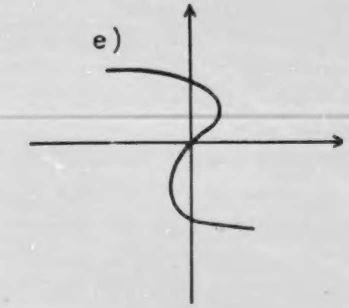
a)



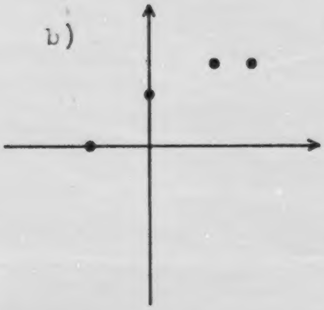
c)



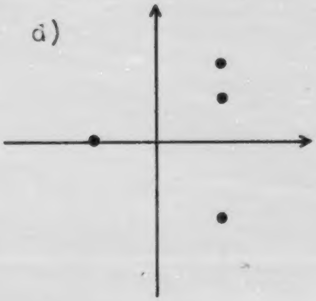
e)



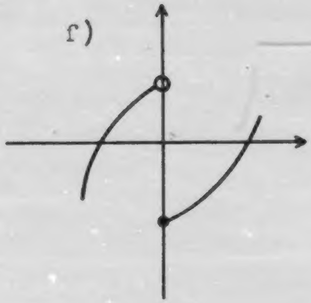
b)



d)



f)



3. What is the constant function whose graph passes through (5, 2)?
4. For what values of a, b, and c, will $f: x \longrightarrow ax^2 + bx + c$ be a constant function?
5. What is the constant function whose graph passes through the intersection of $L_1: y = 3x - 2$, and $L_2: 3y - 4x + 5 = 0$?
6. At what point do $L_1: y = ax + 4$, and $L_2: y = 5x + b$, intersect? Do they always intersect?
7. Write the linear functions f_1 and f_2 whose graphs intersect the x-axis at $P(-3, 0)$ at angles of 45° and -45°, respectively.
8. If $10x + y - 7 = 0$, what is the decrease in y as x increases from 500 to 505? What is the increase in x as y decreases from -500 to -505?
9. Write the equation of the line through (0, 0) which is parallel to the line through (2, 3) and (-1, 1).
10. Write the equation of the line which passes through the intersection of $L_1: y = 6x + k$, and $L_2: y = 5x + k$, and has slope 5/6.
11. Write the equation of the line which is the locus of points equidistant from $L_1: 6x + 3y - 7 = 0$ and $L_2: y = -2x + 3$.
12. Write the equation of the line through (8, 2) which is perpendicular to (has a slope which is the negative reciprocal of the slope of) $L_1: 2y = x + 3$.
13. In a manufacturing process, a certain machine requires 10 minutes to warm up and then produces y parts every t hours. If the machine has produced 20 parts after running 1/2 hour and 95 parts after running 1 3/4 hours, find a function f such that $y = f(t)$, and give the domain of f.
14. If ABCD is a parallelogram with vertices at A(0, 0), B(8,0), C(12, 7), and D(4, 7), find
    a) the equation of the diagonal AC;
    b) the equation of the diagonal BD;
    c) the point of intersection of the diagonals.
15. Repeat Problem 14, using parallelogram ABCD with vertices at $A(0, 0)$, $B(x_1, 0)$, $C(x_2, y_2)$, and $D(x_2 - x_1, y_2)$.
16. Given the constant functions $f: x \longrightarrow a$, $g: x \longrightarrow b$, and $h: x \longrightarrow c$, determine the compound functions $f(gh)$ and $f(hg)$. Does this

result indicate that gh = hg?

17. Find an inverse of the linear function f: $x \longrightarrow mx + b$.

18. Find a function f such that ff = f.

19. Sketch a graph of:
    a) $f(x) = \frac{|x|}{x}$
    b) $|x| + |y| = 1$
    c) $y = |x - 1| - |x + 1|$

20. If $f(x) = 2x - 5$ and $g(x) = 3x + k$, determine k so that fg = gf.

21. If $f(x) = x^2$ and $g(x) = \sqrt{16 - x^2}$, find the domains of fg and gf.

Chapter 2

POLYNOMIAL FUNCTIONS

2-1. Introduction and Notation

In this chapter we shall be concerned with functions that are defined by expressions of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where n is a positive integer or zero, the coefficients $a_i (i=0,1,2,3,\dots,n)$ are real numbers, and $a_n \neq 0$. Such expressions are called polynomials, and the functions which they define are called polynomial functions. The number n is called the degree of the polynomial.

Examples:

(1) $5x^3 - 3x^2 + x + 10$ is a polynomial of degree 3.

(2) $f: x \longrightarrow x^4 - \frac{3}{8} x^2$ is a polynomial function of degree 4.

In the preceding chapter we discussed polynomial functions of the types

$$f: x \longrightarrow c$$

and

$$f: x \longrightarrow mx + b, \ m \neq 0,$$

which are called constant and linear functions, respectively. It is natural to turn next to quadratic functions, that is, polynomial functions of degree 2,

$$f: x \longrightarrow ax^2 + bx + c, \ a \neq 0,$$

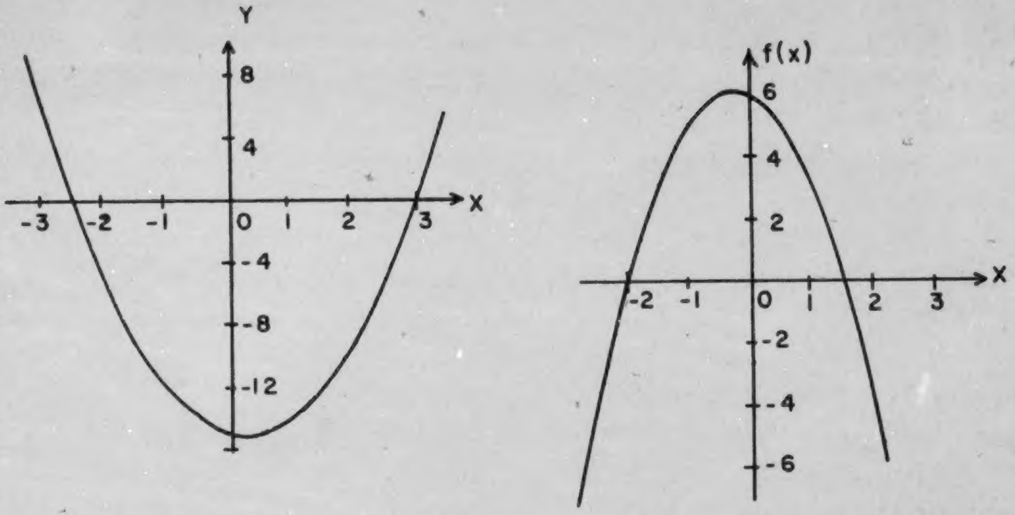
where a, b, and c are constants. Such functions occur frequently, as, for example, in the study of the flight of projectiles, and are familiar to you. The most common way to describe functions of this kind is by equations such as

$$y = 2x^2 - x - 15, \text{ or} \qquad (1)$$

$$f(x) = -2x^2 - x + 6. \qquad (2)$$

Each of these functions can be represented pictorially by a graph (See Figures 2-1).

An immediate concern is the location of the points, if any, where the graphs of these functions intersect the horizontal axis, that is, the points $(x, f(x))$ where $f(x) = 0$. We have at hand



$y = 2x^2 - x - 15$ $\qquad$ $f(x) = -2x^2 - x + 6$

Figure 2-1. Graphs of quadratic functions.

the means of doing this, namely the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

Applying this formula to Equation (1), we learn that $y = 0$ when $x = 3$ or when $x = -\frac{5}{2}$. Using function notation, we can say:

If $f: x \longrightarrow 2x^2 - x - 15$, then $f(3) = 0$ and $f(\frac{-5}{2}) = 0$. The numbers 3 and $\frac{-5}{2}$ in the domain of the function are mapped onto 0 by f, and hence are called the zeros of f.

Definition 2-1. Let f be a function. If a is a number in the domain of f with the property that $f(a) = 0$, then a is called a zero of f.

The set of all zeros of a function f is the set of all x such that $f(x) = 0$; that is,

$$\text{the set of zeros of } f = \{x: f(x) = 0\}.$$

This is just another way of saying that the zeros of f are the roots or solutions of the equation $f(x) = 0$.

We already know how to find the zeros of polynomial functions of the first and second degree.

If $f: x \longrightarrow mx + b$, $m \neq 0$, then $f(-\frac{b}{m}) = 0$.

If $f: x \longrightarrow ax^2 + bx + c$, $a \neq 0$, then $f(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}) = 0$.

Upon examining these solutions, mathematicians noticed that the zeros are expressed in terms of the coefficients by formulas involving only the rational operations (addition, subtraction, multiplication, division) and the extraction of roots of numbers, and believed that it might be possible to express the zeros of functions of higher degree than the quadratic in the same manner. In the first half of the sixteenth century such formal expressions for the zeros of the third and fourth degree polynomial functions were obtained by Italian mathematicians. Unfortunately, these formulas are too complicated to be of practical value in mathematical analysis. Mathematicians usually find it easier even in theoretical questions to work with the polynomial rather than with any explicit expression for the zeros.

For the better part of two centuries, mathematicians tried to discover methods for solving equations of the fifth and higher degrees by formulas analogous to those for the quadratic, cubic, and quartic (also called biquadratic) equations. The attempt was doomed to failure. In 1824, a young Norwegian mathematician, Niels Henrik Abel, proved that it is generally impossible to express the zeros of a polynomial function of degree higher than four in the desired way. This does not mean that mathematicians are unable to obtain any formal solutions for equations of higher degree, but only that it is impossible to obtain general formulas for solutions in terms of the rational operations and the extractions of roots alone. Although the history of the problem seems to end in failure, the fact is that the methods developed by Abel and his equally young French contemporary, Evariste Galois, have found the widest and most useful applications in fields remote from the problem they considered. It is often that way in mathematics; the methods used to attack a problem frequently have value long after the problem itself has lost its special significance. (There are a number of interesting accounts of the historical developments mentioned above; see the references listed in the bibliography at the end of this chapter.)

Even though we shall not develop any general formulas for the zeros of a polynomial function, we shall be able to obtain a great deal of useful information about them. In particular, there are many methods for determining the zeros to any desired decimal accuracy, and we shall examine some of these.

Before beginning the general discussion of polynomial functions of degrees greater than two, we must give some attention to the notation we are to use. Any polynomial function will be denoted by a lower case letter, commonly f, although we shall occasionally need additional letters such as g, q, r, etc. If we wish to emphasize the degree of the function, we shall indicate it by an appropriate subscript. Thus f_3 will indicate a polynomial function of degree 3, while f_n and f_0 will indicate polynomial functions of degrees n and 0, respectively. The coefficients of the polynomial will be written as a_i with the subscript i equal to the exponent of the power of x in that term. Thus, in this notation we write:

$f_1: x \rightarrow a_1x + a_0,\ a_1 \neq 0$, for the linear function;

$f_2: x \rightarrow a_2x^2 + a_1x + a_0,\ a_2 \neq 0$, for the general quadratic function;

$f_n: x \rightarrow a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,\ a_n \neq 0$, for the general polynomial function of degree n. The three dots in this formula are the conventional representation of the omitted terms of the polynomial.

Definition 2-2. A polynomial function of degree n, where n is a positive integer or zero, is an association

$f_n: x \rightarrow a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,\ a_n \neq 0$, where the domain is the set R of all real numbers, and the range is the set (or a subset) of real numbers

$\{y: y = f(x),\ x \in R\}$.

Example 1. The function $f_2: x \rightarrow x^2 + 1$ is a polynomial of degree 2 with range $\{y: y \geq 1\}$. (Technically, the word "polynomial" is the name of the expression -- in this case $x^2 + 1$ -- which describes the polynomial function. It is common practice, however, to use the word "polynomial" in the place of "polynomial function" when the context makes it clear that we are really talking about the function.)

While the coefficients a_i $(i = 0, 1, \ldots, n)$ in general stand for any real numbers, in our examples and exercises they will usually represent integers. Near the end of the chapter we shall extend the domain and range of polynomial functions and the coefficients of $f(x)$ to the complex number system.

Before concluding this section, we note that the degree of a polynomial function is uniquely defined. That is, if for all real x a given polynomial function can be expressed as

$$x \rightarrow a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

and also as

$$x \rightarrow b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0,$$

then n must equal m, and the corresponding coefficients must also be equal,

$$a_i = b_i \qquad (i = 0, 1, 2, \ldots, n).$$

We shall not prove this, but we state it for the sake of completeness.

If the degree of a polynomial function is 0, then the function is

$$f: x \rightarrow a_0,\ a_0 \neq 0,$$

which we recognize to be a constant function. It is useful for certain purposes to consider the special constant function that maps every real number into 0,

$$f: x \rightarrow 0,$$

as a polynomial function. $f(x)$ is then called the zero polynomial (0 polynomial). The zero polynomial has no degree and is not a polynomial of degree zero. To summarize,

$f: x \rightarrow a_0$, $a_0 \neq 0$, is a polynomial function of degree 0;

$f: x \rightarrow 0$ is the 0 polynomial, to which we assign no degree.

2-2. Evaluation of $f(x)$ at $x = c$.

Most of our work with polynomial functions will be concerned with two related problems:

Problem 1. Given a function f and any number x in its domain, find $f(x)$.

Problem 2. Given a function f and any number y in its range, find $\{x : f(x) = y\}$.

In later sections we shall study the second, and harder, of these two problems. In this section we study Problem 1. To graph polynomial functions and find the solutions of polynomial equations, it is important to evaluate a given $f(x)$ for different values of x. For example, to graph

$$f: x \longrightarrow 3x^3 - 2x^2 + x - 6,$$

we may want the values $f(x)$ at $x = 0, 1, 2, 3$, etc. Of course, we may obtain these values by direct substitution, doing all of the indicated multiplications and additions. For most values this is tedious. Fortunately, there is an easier way which we shall call synthetic substitution. To understand the method, we shall analyze a few easy examples.

Example 1. Find the value of

$$f(x) = 2x^2 - x + 3 \qquad \text{at } x = 4.$$

We write

$$f(x) = (2x - 1)x + 3.$$

When $x = 4$, this becomes

$$[2(4) - 1]4 + 3 = 31.$$

Note that to evaluate our expression, we can

a) Multiply 2 (the coefficient of x^2) by 4 and add this product to -1 (the coefficient of x);

b) Multiply the result of (a) by 4 and add this product to to 3 (the constant term).

Example 2. Find $f(3)$, given

$$f(x) = 2x^3 - 3x^2 + 2x + 5.$$

$f(x)$ may be written

$$(2x^2 - 3x + 2)x + 5$$

or $$[(2x - 3)x + 2]x + 5.$$

To find the value of this expression when $x = 3$, we may start with the inside parentheses and

a) Multiply 2 (the coefficient of x^3) by 3 and add this product to -3 (the coefficient of x^2);

b) Multiply the result of (a) by 3 and add this product to 2 (the coefficient of x);

c) Multiply the result of (b) by 3 and add this product to 5 (the constant term).

The result is $f(3) = 38$.

These steps can be represented conveniently by a table whose first row consists of the coefficients of the successive powers of x in descending order: (The number at the far right is the particular value of x being substituted.)

$$\begin{array}{cccc|l} 2 & -3 & 2 & 5 & \underline{3} \\ & \nearrow(2 \cdot 3) = 6 & \nearrow(3 \cdot 3) = 9 & \nearrow(11 \cdot 3) = 33 & \\ \hline 2 & 3 & 11 & 38 & \end{array}$$

When this tabular arrangement is used, we proceed from left to right. We start the process by rewriting the first coefficient, 2, in the third row. Each entry in the second row is 3 times the entry in the third row of the preceding column. Each entry in the third row is the sum of the two entries above it. We note that the result, 38, can be checked by direct substitution.

Now let us consider the general cubic polynomial

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0,\ a_3 \neq 0.$$

When $x = c$, we have

$$f(c) = a_3c^3 + a_2c^2 + a_1c + a_0,$$

which may be written

$$f(c) = [(a_3c + a_2)c + a_1]c + a_0.$$

Again the steps employed in the procedure can be represented in tabular form:

$$\begin{array}{cccc|l} a_3 & a_2 & a_1 & a_0 & \underline{c} \\ & a_3c & (a_3c + a_2)c & [(a_3c + a_2)c+a_1]c & \\ \hline a_3 & a_3c + a_2 & (a_3c + a_2)c + a_1 & f(c) & \end{array}$$

As in earlier examples, the number being substituted is written to the right of the entire array.

Let us do a few more examples.

Example 3. Given $f(x) = 3x^3 - 2x^2 + x - 6$, determine $f(2)$.

$$\begin{array}{rrrr|l} 3 & -2 & 1 & -6 & \underline{2} \\ & 6 & 8 & 18 & \\ \hline 3 & 4 & 9 & 12 & \end{array}$$

Now 12 is the result sought, namely f(2). This may be checked by direct substitution:

$$f(2) = 3(2)^3 - 2(2)^2 + 2 - 6 = 24 - 8 + 2 - 6 = 12.$$

Example 4. Given $f(x) = x^4 - 3x^2 + 2x - 5$, determine f(3). Note that $a_3 = 0$ and that this number must be written in its appropriate place as one of the detached coefficients in the first row.

$$\begin{array}{rrrrr|l} 1 & 0 & -3 & 2 & -5 & \underline{3} \\ & 3 & 9 & 18 & 60 & \\ \hline 1 & 3 & 6 & 20 & 55 & \end{array}$$

Thus, f(3) = 55, which, as before, may be checked by direct substitution.

With a little care and practice, the second line in the above work can often be omitted when c is a small integer.

Example 5. Given $f(x) = x^4 - x^3 - 16x^2 + 4x + 48$, evaluate f(x) for x = -3, -2, -1, 0, 1, 2, 3, 4, 5.

Solution. We detach the coefficients. In order to avoid confusion, it is sometimes convenient to write them down at the bottom of a sheet of scratch paper and slide this down, covering at each step the work previously done. As suggested above, we omit the second line in each evaluation and write the value of x we are using adjacent to the answer. The results appear in Table 2-1.

The last two columns now become a table of f(x) and x. Note that the row that corresponds to x = 0 has the same entries as the coefficient row. Do you see why?

Table 2-1

Computation by Synthetic Substitution

Coefficients						
1	-1	-16	4	48		
1	-4	- 4	16	0	-3
1	-3	-10	24	0	-2
1	-2	-14	18	30	-1
1	-1	-16	4	48	0
1	0	-16	-12	36	1
1	1	-14	-24	0	2
1	2	-10	-26	-30	3
1	3	- 4	-12	0	4
1	4	4	24	168	5
				$f(x)$	x

The method described and illustrated above is often called synthetic substitution or synthetic division in algebra books. The word "synthetic" literally means "put together," so you can see how it is that "synthetic substitution" is appropriate here; in Section 2-4, you will see why the process is also called "division." The method gives a quick and efficient means of evaluating $f(x)$, and we are now able to plot the graphs of polynomials more easily than would be the case if the values of $f(x)$ had to be computed by direct substitution.

Exercises 2-2

Evaluate the following polynomials for the given values of x.

1. $f(x) = x^4 + x - 3$; $x = -2, 1, 3.$
2. $f(x) = x^2 - 3x^3 + x - 2$; $x = -1, -3, 0, 2, 4.$
3. $g(x) = 3x^3 - 2x^2 + 1$; $x = \frac{1}{2}, \frac{1}{3}, 2.$

4. $r(x) = 6x^3 - 5x^2 - 17x + 6; \qquad x = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, 2.$
5. $s(x) = 6x^3 - 29x^2 + 37x - 12; \qquad x = 0, 1, 2, 3, 4.$
6. If $f(x) = 2x^3 - kx^2 + 3x - 2k$, for what value of k will $f(2) = 4$?
7. Evaluate $3x^4 - 97x^3 + 35x^2 + 8x + 2$ for $x = 1/3$
   a) directly,
   b) by synthetic substitution.
8. Evaluate $x^{10} - 4x^3 + 10$ for $x = 2$
   a) directly,
   b) by synthetic substitution.

2-3. Graphs of Polynomial Functions

As stated at the end of Section 2-2, synthetic substitution greatly simplifies the problem of graphing.

Example. Plot the graph of the polynomial function

$$f: x \longrightarrow 2x^3 - 3x^2 - 12x + 13$$

We prepare a table of values of x and $f(x)$ by synthetic substitution, and then plot the points whose coordinates $(x, f(x))$ appear in the table. The work is shown in Table 2-2.

Table 2-2

Finding Coordinates $(x, f(x))$ by Synthetic Substitution

Coefficients				
2	-3	-12	13	
2	-9	15	-32	-3
2	-7	2	9	-2
2	-5	- 7	20	-1
2	-3	-12	13	0
2	-1	-13	0	1
2	1	-10	- 7	2
2	3	- 3	4	3
2	5	8	45	4
			f(x)	x

From the table we observe that the points (x, f(x)) to be plotted are (-3, -32), (-2, 9), (-1, 20), etc. These points are located on a rectangular coordinate system as shown in Figure 2-3a. Note that we have chosen different scales on the axes for convenience in plotting.

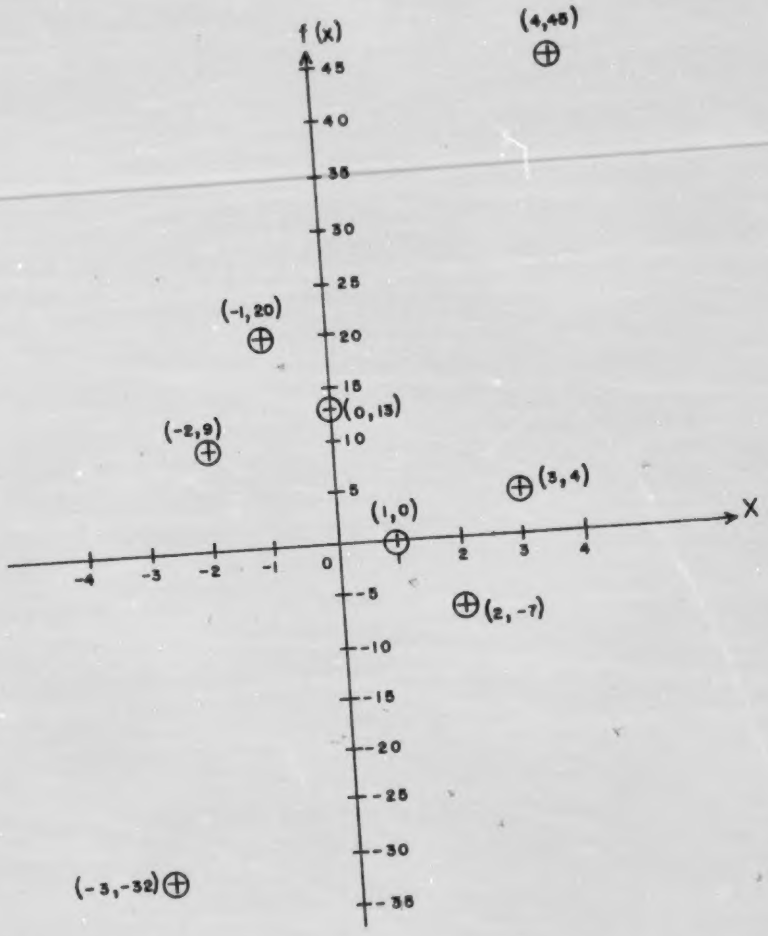


Figure 2-3a.
Points on the graph of f: $x \longrightarrow 2x^3 - 3x^2 - 12x + 13$.

Now the problem is how best to draw the graph. An inspection of the given polynomial $2x^3 - 3x^2 - 12x + 13$, shows that for any real value of x a value of $f(x)$ exists. We shall assume that the graph is a continuous curve with no breaks or holes in it. But the question still remains whether the points we have already plotted are sufficient to give us a fairly accurate picture of the graph, or whether there may be hidden "peaks" and "valleys" not shown thus far. We are not in a position to answer this question categorically at present, but we can shed further light on it by plotting more points between those already located. By use of fractional values of x and the method of synthetic substitution, Table 2-2 is extended as shown in Table 2-3.

Table 2-3

Additional Coordinates of Points on the Graph of f

Coefficients				
2	-3	-12	13	
2	-8	8	-7	$-\frac{5}{2}$
2	-6	-3	$\frac{35}{2}$	$-\frac{3}{2}$
2	-4	-10	18	$-\frac{1}{2}$
2	-2	-13	$\frac{13}{2}$	$\frac{1}{2}$
2	0	-12	-5	$\frac{3}{2}$
2	2	-7	$-\frac{9}{2}$	$\frac{5}{2}$
2	4	2	20	$\frac{7}{2}$
			$f(x)$	x

When we fill in these points on the graph, it appears that if we connect the points by a smooth curve, we ought to have a reasonably accurate picture of the graph of f in the interval from -3 to 4. This is shown in Figure 2-3b.

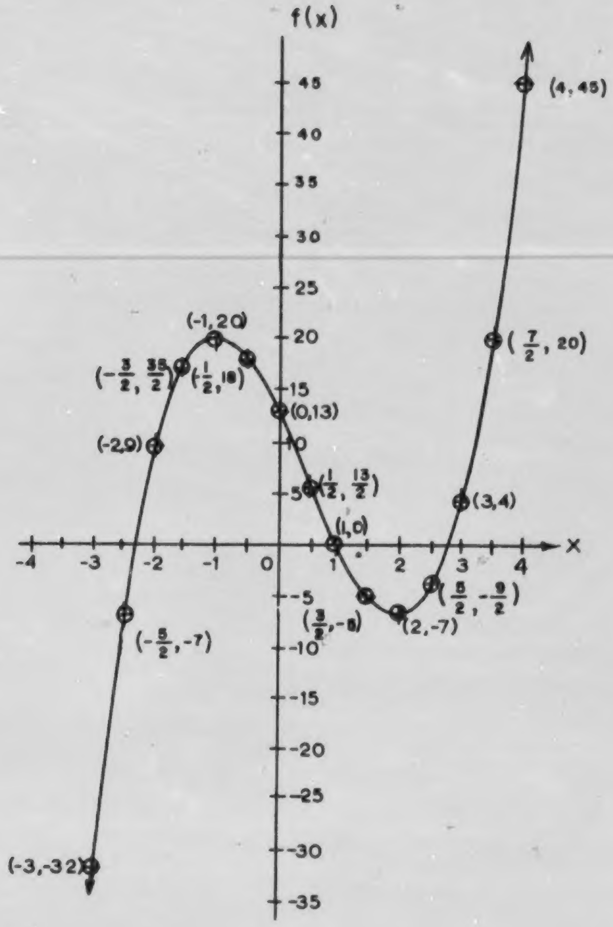


Figure 2-3b.
Graph of $f: x \longrightarrow 2x^3 - 3x^2 - 12x + 13$.

What about the shape of the graph outside of this interval, in particular, for large values of $|x|$? The easiest way to answer this question is to look at $f(x)$ for $x = 10, 100, -10, -100$, etc. - values which are easy to find by direct substitution. The function under consideration is

$$f: x \longrightarrow 2x^3 - 3x^2 - 12x + 13,$$

and

$$f(10) = 2000 - 300 - 120 + 13,$$

$$f(100) = 2{,}000{,}000 - 30{,}000 - 1200 + 13.$$

It will be observed that even when x is no greater than 10, the term of highest degree is a much larger number than any of the other terms, and for $x = 100$ the difference is much greater. Hence, $f(10)$ and $f(100)$ are large positive numbers, and the points on the graph corresponding to them are far above the x-axis.

Likewise, $f(-10) = -2000 - 300 + 120 + 13$

and $f(-100) = -2{,}000{,}000 - 30{,}000 + 1200 + 13.$

Again, the value of the term of highest degree clearly dominates the other terms, and $f(-10)$ and $f(-100)$ are negative numbers that correspond to points on the graph far below the x-axis.

Writing the given polynomial in a factored form, $2x^3 - 3x^2 - 12x + 13 = 2x^3(1 - \frac{3}{2x} - \frac{6}{x^2} + \frac{13}{2x^3})$, may help to show why the term $2x^3$ dominates all other terms for large $|x|$. The fractions containing x in the denominator decrease numerically as $|x|$ increases, so that for sufficiently large values of $|x|$, the expression in parentheses has a value close to 1.

By this kind of reasoning we can deduce that for any polynomial the term of highest degree will dominate all other terms for large values of $|x|$. This means that the sign of $f(x)$ will agree with the sign of the term of highest degree for large $|x|$, and hence the graph of f will lie above or below the x-axis according as the value of this term is positive or negative. Also one can reason that the n^{th} degree polynomial behaves like the linear function when $|x|$ is very small.

The question now arises whether the point $(-1, 20)$ is the highest point on the graph between $x = 0$ and $x = -2$, or whether the highest point may actually be a little to the right or left of $(-1, 20)$. By choosing values of x very close to -1, evaluating $f(x)$ for these values of x, and comparing them with $f(-1)$, we decrease our uncertainty about the location of the highest point, but we cannot assert that (-1, 20) is the highest point in this interval. For example, for $x = -0.9$, synthetic substitution gives $f(-0.9) = 19.912$, which is a little less than $f(-1) = 20$. Likewise, $f(-1.1) = 19.908$, which again is a little less than 20.

But we still do not know whether the point (-1, 20) is the highest point between $x = -0.9$ and $x = -1.1$. The answer to this question will be given in Chapter 3, when we develop a method for finding these so-called "maximum" and "minimum" points.

In spite of our uncertainty about its exact shape, the graph does give us information about the zeros of the function. We note that the graph crosses the x-axis at (1, 0); in other words, $f(1) = 0$, and hence, by definition, 1 is a zero of f. Looking further, we see that the graph also crosses the x-axis between $x = -\frac{5}{2}$ and $x = -2$, and again between $x = \frac{5}{2}$ and $x = 3$. Since the abscissa of each intersection with the x-axis is a number for which $f(x) = 0$ and hence by definition a zero of f, we conclude that f has a zero between $-\frac{5}{2}$ and -2 and another between $\frac{5}{2}$ and 3. The graph does not enable us to determine whether these zeros are rational or irrational; this question will be considered in succeeding sections.

Exercises 2-3

Draw the graph of each of the following functions:

1. $f: x \longrightarrow -2x^3 + 3x^2 + 12x - 13$ (Compare this graph with the one in Figure 2-3b. What do you observe?)
2. $f: x \longrightarrow 2x^3 - 12x + 13$
3. $f: x \longrightarrow 2x^3 - 3x^2 - 12x$ (Compare this graph with the one in Figure 2-3b. What do you observe?)
4. $f: x \longrightarrow x^3$ (In this case, direct substitution is faster than synthetic substitution.)
5. $f: x \longrightarrow x^3 + 4$ (How does this graph compare with that of Exercise 4?)
6. $f: x \longrightarrow x^3 - 3x^2 + 4$

7. $f: x \longrightarrow x^4$ (Use generous scales on both axes and draw the graph from $x = -\frac{3}{2}$ to $x = \frac{3}{2}$. Include the points where $x = \pm \frac{1}{2}$ and $x = \pm \frac{3}{4}$)

8. $f: x \longrightarrow x^4 - 2x^3 - 5x^2 + 6x$

2-4. Remainder and Factor Theorems

Momentarily we shall turn away from graphing and take another look at the process we described in Section 2-2, in order to develop two theorems that will be useful in finding the zeros of polynomial functions. The synthetic substitution used to determine $f(2)$, given

$$f: x \longrightarrow x^3 - 7x^2 + 3x - 2,$$

will be the basis for this development, so let us examine it closely.

$$\begin{array}{rrrr|l} 1 & -7 & 3 & -2 & \underline{2} \\ & 2 & -10 & -14 & \\ \hline 1 & -5 & -7 & -16 & \end{array}$$

We rewrite the first row in the synthetic substitution as the given polynomial (by restoring the powers of x), and then attach the same power of x to each entry in a given column. Thus we obtain

$$\begin{array}{rrrr} 1x^3 & -7x^2 & +3x & -2 \\ & 2x^2 & -10x & -14 \\ \hline 1x^3 & -5x^2 & -7x & -16 \end{array}$$

The polynomial in the third row is the sum of the two preceding polynomials. Since $f(x) = x^3 - 7x^2 + 3x - 2$ and $f(2) = -16$, the above addition can be written

$$f(x) + 2x^2 - 10x - 14 = x^3 - 5x^2 - 7x + f(2).$$

By factoring, we may write

$$f(x) + 2(x^2 - 5x - 7) = x(x^2 - 5x - 7) + f(2).$$

Solving for $f(x)$, we have

$$f(x) = x(x^2 - 5x - 7) - 2(x^2 - 5x - 7) + f(2),$$

or $$f(x) = (x - 2)(x^2 - 5x - 7) + f(2).$$

The form of this expression may look familiar. It is, in fact, an example of the division algorithm:

Dividend = (Divisor)(Quotient) + Remainder.

In our example, if $(x - 2)$ is the divisor, then

$$q(x) = x^2 - 5x - 7$$

is the quotient, and $f(2)$ is the remainder. This result may be generalized. It is of sufficient importance to be stated as a theorem.

Theorem 2-1. Remainder Theorem. If $f(x)$ is a polynomial of degree $n > 0$ and if c is a number, then the remainder in the division of $f(x)$ by $x - c$ is $f(c)$. That is,

$$f(x) = (x - c)q(x) + f(c),$$

where the quotient $q(x)$ is a polynomial of degree $n - 1$.

Proof: We shall prove the theorem only in the case of the general cubic polynomial,

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

Following the pattern of the previous example, to determine $f(c)$ we write the synthetic substitution

$$\begin{array}{ccccc} a_3 & a_2 & a_1 & a_0 & \underline{|c} \\ & a_3c & (a_3c + a_2)c & (a_3c^2 + a_2c + a_1)c & \\ \hline (a_3) & (a_3c + a_2) & (a_3c^2 + a_2c + a_1) & (a_3c^3 + a_2c^2 + a_1c + a_0) & \end{array}$$

As before, writing in the appropriate powers of x, we get

$$\begin{array}{cccc} a_3x^3 & + a_2x^2 & + a_1x & + a_0 \\ & + a_3cx^2 & + (a_3c + a_2)cx & + (a_3c^2 + a_2c + a_1)c \\ \hline a_3x^3 & + (a_3c + a_2)x^2 & + (a_3c^2 + a_2c + a_1)x & + (a_3c^3 + a_2c^2 + a_1c + a_0) \end{array}$$

We note that the polynomial in the third row is the sum of the two preceding polynomials, that the polynomial in the first row is $f(x)$ and that $(a_3c^3 + a_2c^2 + a_1c + a_0)$ is $f(c)$. Hence we may write

$$f(x) + c[a_3x^2 + (a_3c + a_2)x + (a_3c^2 + a_2c + a_1)] =$$
$$x[a_3x^2 + (a_3c + a_2)x + (a_3c^2 + a_2c + a_1)] + f(c).$$

Thus we have

$$f(x) = (x - c)[a_3x^2 + (a_3c + a_2)x + (a_3c^2 + a_2c + a_1)] + f(c)$$

or $$f(x) = (x - c)q(x) + f(c). \quad \text{q.e.d.}$$

The process is the same for higher degree polynomials. It gives

$$f(x) = (x - c)q(x) + f(c),$$

where $q(x)$ is a polynomial of degree $n - 1$.

If the remainder $f(c)$ is zero, then the divisor $x - c$ and the quotient $q(x)$ are factors of $f(x)$. Hence we have a second theorem:

Corollary 2-1-1. Factor Theorem. If c is a zero of a polynomial function f of degree $n > 0$, then $x - c$ is a factor of $f(x)$, and conversely.

Proof. We know from Theorem 2-1 that there exists a polynomial $q(x)$ of degree $n - 1$ such that

$$f(x) = (x - c)q(x) + f(c).$$

If c is a zero of f, then $f(c) = 0$ and

$$f(x) = (x - c)q(x).$$

Hence $x - c$ is a factor of $f(x)$, by definition.

Conversely, if $x - c$ is a factor of $f(x)$, then by definition there is a polynomial $q(x)$ such that

$$f(x) = (x - c)q(x).$$

For $x = c$, we obtain

$$f(c) = (c - c)q(c) = 0,$$

and hence c is a zero of f. q.e.d.

Example 1. Find the quotient and remainder if

$$f(x) = 2x^3 - 6x^2 + x - 5$$

is divided by $x - 3$.

Solution.

$$\begin{array}{rrrr|l} 2 & -6 & 1 & -5 & 3 \\ & 6 & 0 & 3 & \\ \hline 2 & 0 & 1 & -2 & \end{array}$$

Hence,

$$q(x) = 2x^2 + 1,$$

$$f(3) = -2,$$

and

$$2x^3 - 6x^2 + x - 5 = (x - 3)(2x^2 + 1) - 2.$$

Example 2. Show that $x - 6$ is a factor of

$$f(x) = x^3 - 6x^2 + x - 6,$$

and find the associated $q(x)$.

Solution.

$$\begin{array}{rrrr|l} 1 & -6 & 1 & -6 & 6 \\ & 6 & 0 & 6 & \\ \hline 1 & 0 & 1 & 0 & \end{array}$$

Here, $f(6) = 0$, $q(x) = x^2 + 1$, and

$$f(x) = (x - 6)(x^2 + 1).$$

In testing for the divisibility of a polynomial by $mx + b$,

$m \neq 0$, we write

$$mx + b = m(x + \frac{b}{m}) = m[x - (- \frac{b}{m})]$$

and see whether $f(- \frac{b}{m}) = 0$. By the Factor Theorem, $mx + b$ is a factor of $f(x)$ if and only if $f(- \frac{b}{m}) = 0$. (Note that $- \frac{b}{m}$ is the root of $mx + b = 0$.)

In applying the Factor Theorem, it may sometimes be easier to compute $f(c)$ by direct substitution, rather than by the method of synthetic substitution. Thus, to show that $x - 1$ is a factor of

$$f(x) = 2x^{73} - x^{37} - 1,$$

we note that $\qquad f(1) = 2 - 1 - 1 = 0.$

Evaluating $f(1)$ by the synthetic substitution method would take considerably longer!

At this point you may wonder what to do when confronted with a polynomial such as

$$8x^4 - 28x^3 - 62x^2 + 7x + 15$$

which you might like to factor. Note that the Factor Theorem is only a testing device. It does not locate zeros of polynomial functions. Methods, other than blind guessing, for doing this will be developed in the next sections.

Exercises 2-4

1. Find $q(x)$ and $f(c)$ so that $f(x) = (x - c)q(x) + f(c)$ if
   a) $f(x) = 3x^3 + 4x^2 - 10x - 15$ and $c = 2$
   b) $f(x) = x^3 + 3x^2 + 2x + 12$ and $c = -3$
   c) $f(x) = -2x^4 + 3x^3 + 6x - 10$ and $c = 3$
   d) $f(x) = 2x^3 - 3x^2 + 5x - 2$ and $c = \frac{1}{2}$
2. Find the quotient and remainder when
   a) $x^3 + 4x^2 - 7x - 3$ is divided by $x - 2$
   b) $x^3 + 3x^2 - 4$ is divided by $x + 2$
   c) $3x^3 + 4x^2 - 7x + 1$ is divided by $3x - 2$
3. If $f_n(x)$ is divided by $g_m(x) \neq 0$ so that a quotient $q(x)$ and a remainder $r(x)$ are obtained, what is the degree of $q(x)$? of $r(x)$?

4. In Exercises 1 to 5 of Section 2-2 do any of the polynomials have linear factors? If so, find them.

5. If $f(x) = x^3 + 4x^2 + x - 6$, determine $f(x)$ at $x = 3, 2, 1, 0, -1, -2, -3$. What are the factors of $f(x)$?

6. If $f(x) = 2x^3 + x^2 - 5x + 2$, determine $f(x)$ at $x = -2, -1, 0, 1, 2$, and $\frac{1}{2}$. What are the factors of $f(x)$?

7. If $f(x) = x^3 + 3x^2 - 12x - k$, find k so that $f(3) = 9$.

8. If $f(x) = x^3 - x^2 + kx - 12$, find k so that $f(x)$ is exactly divisible by $x - 3$.

9. If $f(x) = ax^5 + ax^4 + 13x^3 - 11x^2 - 10x - 2a$, and if $f(-1) = 0$, what is $f(1)$?

10. The quadratic formula enables us to find the zeros of any quadratic function and hence, by the Factor Theorem, to write any quadratic expression in factored form as the product of two first-degree polynomials with complex coefficients (real or imaginary). For example,

$$x^2 - 4x + 1 = (x - 2 - \sqrt{3})(x - 2 + \sqrt{3}),$$

since the roots of $x^2 - 4x + 1 = 0$ are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Write each of the following in factored form over the complex numbers:

a) $2x^2 + 7x - 15$
b) $x^2 - x - 1$
c) $x^2 + 4$
d) $x^2 - 6x + 13$
e) $x^3 - 5x$
f) $2x^2 - 3x + 2$ [Answer: $2(x - \frac{3 + i\sqrt{7}}{4})(x - \frac{3 - i\sqrt{7}}{4})$]
g) $9x^2 + 6x + 5$
h) $2x^2 - 4x + 1$

2-5. Locating Zeros of Polynomial Functions

As has been pointed out earlier, our primary objective in this chapter is to study some methods for finding the zeros of polynomial functions, or, in other words, for solving equations of the form $f(x) = 0$, where $f(x)$ is a polynomial.

If we are confronted with a particular polynomial equation

$f(x) = 0$ and a particular number c, we can easily determine -- by direct or synthetic substitution -- whether or not this number c is a solution. But this technique does not tell us how to find zeros of polynomial functions.

You already know how to solve linear equations and quadratic equations. In fact, you know simple formulas for doing this. In the sixteenth century, attempts were made to find formulas for solving equations of higher degree. Although a few results were obtained, it was found later that to seek formulas is not the best way to approach the problem.

It may surprise you to learn that the best way to solve equations of higher degree is to guess at the solutions. To be sure, it is not at all wise to guess blindly. The purpose of this section and the next is to examine some methods that will enable you to guess intelligently.

From your experience in drawing graphs, you already have a method for estimating the approximate values of the zeros of a polynomial function. (Refer back to Section 2-3.) But plotting graphs is time-consuming, and there are better methods. Inherent in the process of preparing a table for graphing, however, is information that helps us to make intelligent guesses about the zeros. This information is contained in the following theorem.

Theorem 2-2. The Location Theorem. If f is a polynomial function and if a and b are real numbers such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of f between a and b.

Geometrically this theorem means that the graph of f from $(a, f(a))$ to $(b, f(b))$ intersects the x-axis in at least one point. Figure 2-5a illustrates this theorem. (The graph in this figure intersects the x-axis in three places -- hence "at least once.")

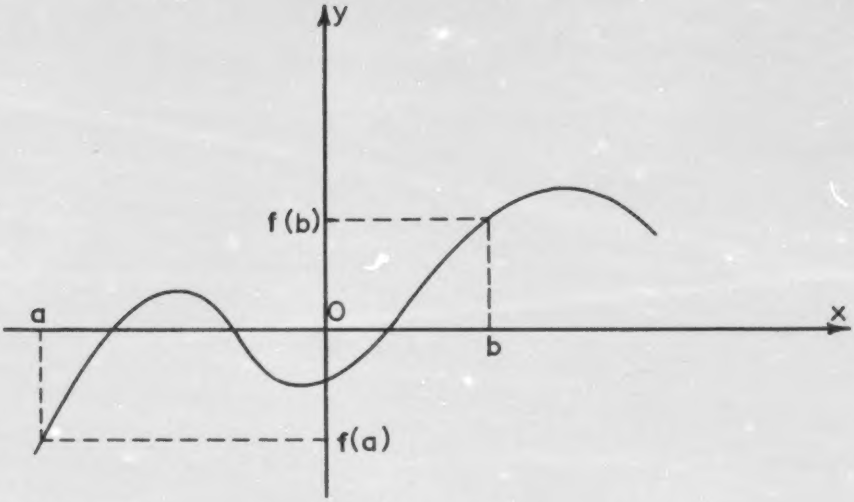


Figure 2-5a.

Illustration of the Location Theorem: f(a) and f(b) are of opposite sign so that f must have at least one zero between a and b.

We shall accept the Location Theorem without proof, first because its proof requires a sequence of theorems that are beyond our reach at this time, and secondly because the result is quite easy to accept intuitively. If the graph is below the x-axis at one point and above it at another, it must cross the x-axis somewhere in between. The crux of the proof consists in showing that the graph of any polynomial function f from $x = a$ to $x = b$ is continuous -- that is, it has no gaps.

Figure 2-5b shows that if, in the Location Theorem, f were not a polynomial function, the conclusion would not necessarily be correct. The curve lies sometimes below and sometimes above the x-axis, yet does not intersect it; however, the graph is not continuous.

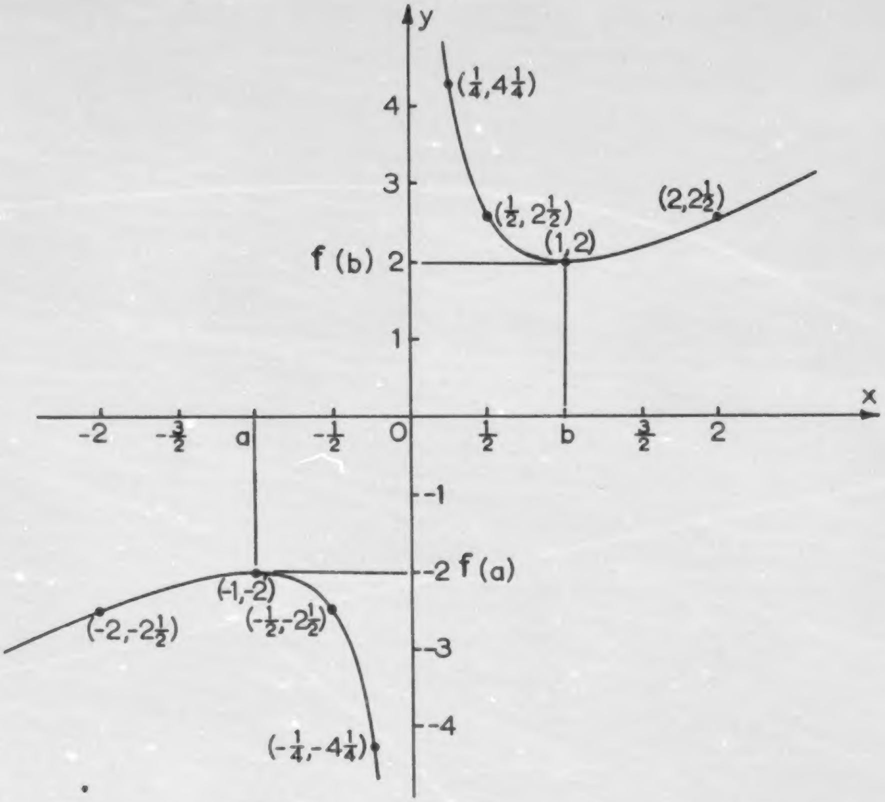


Figure 2-5b. Graph of $y = x + 1/x$.

Example 1. Given that the polynomial function

$$f: x \longrightarrow 12x^3 - 8x^2 - 21x + 14$$

has three real zeros, locate each of them between two consecutive integers.

Solution. We use the Location Theorem to search for values of $f(x)$ that are opposite in sign. It is convenient to do this in a systematic way by synthetic substitution, setting down the work as in Table 2-4.

The intervals that contain the real zeros of f are indicated by the arrows at the right in the table.

Table 2-4

Locating the Zeros of $f:x \longrightarrow 12x^3 - 8x^2 - 21x + 14$

12	-8	-21	14		
12	-8	-21	14	0	Location of a
					← zero
12	4	-17	- 3	1	
					←
12	16	11	36	2	
12	28	63	203	3	
12	-8	-21	14	0	
12	-20	- 1	15	-1	
					←
12	-32	43	-72	-2	
			f(x)	x	

Answer. The real zeros of f are located between 0 and 1, between 1 and 2, and between -2 and -1.

A few remarks concerning the use of the Location Theorem may be helpful. It is quite possible for f(a) and f(b) to be of the same sign, and yet for f to have zeros between a and b, as illustrated in Figure 2-5c, and the zeros may go undetected.

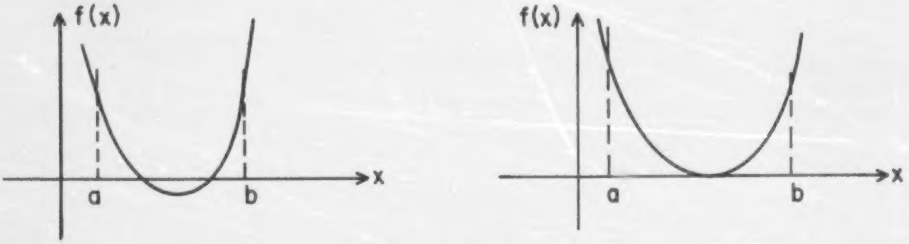


Figure 2-5c.
Graphs with f(a) and f(b) of the same sign, yet with zeros of f between a and b.

Further information to be developed in the remainder of this chapter will be of assistance, but it should be emphasized that the problem of locating the zeros of a polynomial function is essentially a matter of trial.

How far should we extend the table of x and $f(x)$ when searching for the locations of the zeros? This is a very practical question, as illustrated in the next example.

Example 2. Locate the real zeros of $f: x \longrightarrow 2x^3 - x^2 - 2x + 6$.

Solution. The usual procedure is shown in Table 2-5.

Table 2-5
Locating the Zeros of $f: x \longrightarrow 2x^3 - x^2 - 2x + 6$

2	-1	-2	6	
2	-1	-2	6	0
2	1	-1	5	1
2	3	4	14	2
2	5	13	45	3
2	-1	-2	6	0
2	-3	1	5	-1
				←
2	-5	8	-10	-2
2	-7	19	-51	-3
			$f(x)$	x

The Location Theorem tells us that there is at least one real zero between -1 and -2, but what about the other zeros, if any? Later (Section 2-8) we shall show that any polynomial equation of degree $n > 0$ has at most n roots, real or imaginary, and (Section 2-9) that imaginary roots of polynomial equations with real coefficients occur in conjugate pairs. Thus, for the example being considered, there are a number of possibilities: (1) there may be one, two, or three real zeros, all contained in the interval between -1 and -2, (2) two zeros may be imaginary, in which case there is only one real zero, (3) one or two real zeros may be in some other interval of the table between successive integral values of x, or (4) one or two real zeros may be in intervals outside the values of x shown in the table.

Possibility (4) appears unlikely for the simple reason that when we evaluated $f(2) = 14$, all the entries in the corresponding row of Table 2-5 were positive. They will be still greater for greater values of x; the table shows this for $x = 3$, and you can check it yourself for $x = 4$. Thus it appears that for $x > 2$, $f(x)$ must be positive, so that there cannot be a zero of f greater than 2. We prove this, and also that there cannot be a zero of the given polynomial less than -2, by application of the following theorem.

Theorem 2-3. Upper Bound for the Zeros of a Polynomial Function. If a positive number a is substituted synthetically in $f(x)$, where f is a polynomial function, if all the coefficients of $q(x)$ are positive, and if $f(a)$ is also positive, then all the real zeros of f are less than a. We then call a an upper bound for the zeros of f.

Proof. By the Remainder Theorem, $f(x) = (x - a)q(x) + f(a)$. For $x = a$, $f(x) = f(a) > 0$. For $x > a$, by hypothesis, $x - a$, $q(x)$, and $f(a)$ are all positive. Thus, $x \geq a$ is not a zero of f, and all real zeros of f must be less than a.

Now you will see from Table 2-5 that 2 is an upper bound of the zeros of the given polynomial. We really did not need to evaluate $f(3)$.

What about a lower bound for the zeros? Since any negative root of $f(x) = 0$ is a positive root of $f(-x) = 0$, if we find an upper bound for the positive roots of $f(-x) = 0$, its negative will be a lower bound for the negative roots of $f(x) = 0$. Let us apply this test to our example.

From the given polynomial,

$$f(x) = 2x^3 - x^2 - 2x + 6,$$

we find that

$$f(-x) = -2x^3 - x^2 + 2x + 6.$$

Since we are trying to find the roots of the equation, $f(-x) = 0$, it will be less confusing to multiply each member of this equation by -1 in order to have a positive coefficient for the 3rd degree term. This gives the equivalent equation $-f(-x) = 0$, i.e.,

$$2x^3 + x^2 - 2x - 6 = 0.$$

Using synthetic substitution, we obtain the results shown in Table 2-6 for positive values of x.

Table 2-6
Evaluating $-f(-x) = 2x^3 + x^2 - 2x - 6$

2	1	-2	-6	
2	1	-2	-6	0
2	3	1	-5	1
				←
2	5	8	10	2
2	7	19	51	3
			-f(-x)	x

This table tells us two things. First, a positive root of $-f(-x) = 0$ occurs between 1 and 2, which means that a negative root of $f(x) = 0$ occurs between -1 and -2, as previously shown in Table 2-5. Secondly, 2 is an upper bound for the roots of $-f(-x) = 0$, and hence, -2 is a lower bound for the roots of $f(x) = 0$. This is the conclusion which we have been looking for. In actual practice, however, it is unnecessary to evaluate $-f(-x)$ to find a lower bound for the zeros of f. You will notice in Table 2-5 that the synthetic substitution for $x = -2$ gives alternating signs for the coefficients of $q(x)$ and $f(-2)$.

In general, if a negative number a is substituted synthetically in $f(x)$, and if the coefficients of $q(x)$ and the number $f(a)$ alternate in sign, then all of the real zeros of f are greater than a, and a is a lower bound for the zeros.

To conclude Example 2, we have found that 2 is an upper bound and -2 is a lower bound for the real zeros of the given function. Hence, all the real zeros of f are contained in the interval $\{x: -2 < x < 2\}$, and we have found that one zero lies between -1 and -2. For the moment we say no more about the other zeros.

Exercises 2-5

1. Find intervals between consecutive integers that contain the real zeros of f, given that:
   a) $f(x) = x^3 - 3x^2 + 3$
   b) $f(x) = 3x^3 + x^2 + x - 3$
   c) $f(x) = 9 - x - x^2 - x^3$
   d) $f(x) = 3x^3 - 3x + 1$ (Hint: evaluate $f(\frac{1}{2})$.)
   e) $f(x) = 2x^3 - 5x^2 - x + 5$
   f) $f(x) = x^3 - 3x^2 + 6x - 9$
   g) $f(x) = x^4 - 6x^3 + x^2 + 12x - 6$.
2. Determine the range of values of k for which $f(x) = x^3 - 2x^2 + 3x - k$ has at least one real zero between
   a) 0 and 1,
   b) 1 and 2.

2-6. Rational Zeros.

If $f(x)$ is a polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, all of whose coefficients $a_n, a_{n-1}, \ldots, a_0$ are integers, then we may find all rational zeros of f by testing only a finite number of possibilities, as indicated by the following theorem.

Theorem 2-4. Rational Zeros of Polynomial Functions. If the polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \qquad (1)$$

has integer coefficients $a_n, a_{n-1}, \ldots, a_0$, and if f has a rational zero $p/q \neq 0$, $q > 0$, expressed in lowest terms (that is, p and q are integers with no common integer divisor greater than 1), then p is a divisor of a_0 and q is a divisor of a_n.

(Note that in this discussion q is a positive integer and is not to be confused with the polynomial function $q: x \longrightarrow q(x)$.)

Proof. If p/q is a zero of f, then $f(p/q) = 0$.
By Equation (1)

$$f(\frac{p}{q}) = a_n(\frac{p}{q})^n + a_{n-1}(\frac{p}{q})^{n-1} + \ldots + a_1(\frac{p}{q}) + a_0 = 0,$$

or, when cleared of fractions,

$$a_np^n + a_{n-1}p^{n-1}q + \ldots + a_1pq^{n-1} + a_0q^n = 0. \qquad (2)$$

Solving Equation (2) for a_0q^n we obtain

$$\begin{aligned} a_0q^n &= -[a_np^n + a_{n-1}p^{n-1}q + \ldots + a_1pq^{n-1}] \\ &= -p[a_np^{n-1} + a_{n-1}p^{n-2}q + \ldots + a_1q^{n-1}] \\ &= pN, \end{aligned} \qquad (3)$$

where $N = -[a_np^{n-1} + a_{n-1}p^{n-2}q + \ldots + a_1q^{n-1}]$ is an integer. Hence, p divides a_0q^n a whole number, N, of times. We wish to show that p divides a_0. To do this, we appeal to the fundamental Theorem of Arithmetic, that the factorization of positive integers is unique; namely, we note that since p and q have no common integer divisor greater than 1, neither have p and q^n. Hence, all the factors of p are factors of a_0, and p is a factor of a_0.

To prove that q divides a_n, we write Equation (2) in the form

$$a_np^n = -q[a_{n-1}p^{n-1} + \ldots + a_1pq^{n-2} + a_0q^{n-1}]. \qquad (4)$$

Then we reason that since q divides the right-hand side of (4), it divides the number a_np^n. Again, since p and q have no common divisor greater than 1, neither have q and p^n. Hence, all the factors of q are factors of a_n, and q is a factor of a_n. q.e.d.

The foregoing result may be easier to remember if we state it in words: If a fraction in lowest terms is a root of a polynomial equation with integer coefficients, then the numerator of the fraction must divide the constant term of the polynomial, and the denominator must divide the coefficient of the highest power of x. To keep things straight, we can always see how the theorem works for

$$mx + b = 0, \ m \neq 0.$$

The only root is $-b/m$; the numerator $-b$ divides b, while the denominator m divides m.

If the polynomial has fractional coefficients, the theorem can be applied after the polynomial has been multiplied by a non-zero integer to clear of fractions, because the roots of $f(x) = 0$ and the roots of $k[f(x)] = 0$ $(k \neq 0)$ are the same.

Example 1. What are the rational roots of

$$3x^3 - 8x^2 + 3x + 2 = 0?$$

Solution. It is clear that 0 is not a root. If p/q is a rational root, in lowest terms, then

$$p \text{ divides } 2, \quad q \text{ divides } 3.$$

The possibilities are

$$p = \pm 1, \pm 2, \quad q = 1, 3,$$

so that

$$\frac{p}{q} = \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{1}{3}, \text{ or } \pm \frac{2}{3}.$$

We test these one by one and find that the roots of the given equation are 1, 2, and $-\frac{1}{3}$.

(Note that in the statement of Theorem 2-4, we specified $q > 0$, so the possibilities for q are all positive. There is no point in testing both $\frac{1}{3}$ and $\frac{-1}{-3}$.)

Example 2. Find the rational roots of

$$3x^4 - 8x^3 + 3x^2 + 2x = 0.$$

Solution.

$$\begin{aligned} f(x) &= 3x^4 - 8x^3 + 3x^2 + 2x \\ &= x(3x^3 - 8x^2 + 3x + 2). \end{aligned}$$

Now, $f(x) = 0$

if and only if either

$$x = 0$$

or

$$3x^3 - 8x^2 + 3x + 2 = 0. \tag{5}$$

By Example 1, the roots of Equation (5) are 1, 2, and $-\frac{1}{3}$. Adding the root 0, we see that the roots of $f(x) = 0$ are 0, 1, 2, $-\frac{1}{3}$.

Corollary 2-4-1. Integral Zeros. If

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

is a polynomial with integer coefficients, with the constant term $a_0 \neq 0$, and with the coefficient of the highest power of x equal to 1, then the only possible rational zeros of f are integers that divide a_0.

Proof. Suppose p/q (in lowest terms), $q > 0$, is a zero of f. Since $a_0 = f(0) \neq 0$, $p/q \neq 0$. Hence, by Theorem 2-4, p divides a_0 and q divides 1. Therefore, $q = 1$, and $p/q = p$ is an integer that divides a_0. q.e.d.

Example 3. Find the rational zeros of

$$f: x \longrightarrow x^3 + 2x^2 - 9x - 18.$$

Solution. By Corollary 2-4-1, the possible rational zeros are integers that divide -18, namely ± 1, ± 2, ± 3, ± 6, ± 9, ± 18. By trial, the zeros of f are -3, -2, and 3.

Reduction of degree. After we have found one zero of a polynomial function f, we can use a special device to make it easier to find further zeros. By this device, we can cut down the number of possible zeros we have to test, and sometimes we can even use it to help us find certain irrational zeros. We explain this device as follows:

We know from the Factor Theorem (Corollary 2-1-1) that a is a zero of f if and only if there is a polynomial q such that

$$f(x) = (x - a)q(x). \tag{6}$$

Since the product $(x - a)q(x)$ is zero if and only if either $x - a = 0$ or $q(x) = 0$, it follows that the set of zeros of f consists of a together with the set of zeros of q:

$$\{x : f(x) = 0\} = \{x : x = a \text{ or } q(x) = 0\}. \tag{7}$$

Moreover, the degree of q is one less than the degree of f. Thus, if we can find one zero of f, Equations (6) and (7) allow us to reduce the problem of finding the zeros of f to that of finding the zeros of a polynomial q of lower degree. Naturally we may repeat the process, with q in place of f, if we are fortunate enough to find a zero of q, say b. For then we may apply the Factor Theorem to q and write

$$q(x) = (x - b)r(x),$$

and

$$\{x : q(x) = 0\} = \{x : x = b \text{ or } r(x) = 0\}.$$

If we are successful in repeating this reduction until we have a quotient which is either linear or quadratic, we can easily finish the job by solving a linear or quadratic equation.

Example 4. Find all solutions of

$$2x^3 - 3x^2 - 12x + 13 = 0. \tag{8}$$

Solution. We noticed in Section 2-3, Figure 2-3b, that 1 is a solution of Equation (8). Therefore, $x - 1$ is a divisor of

$2x^3 - 3x^2 - 12x + 13$. Performing the division,

$$\begin{array}{rrrr} 2 & -3 & -12 & 13 \\ & 2 & -1 & -13 \\ \hline 2 & -1 & -13 & 0 \end{array} \qquad \lfloor\underline{1}\;.$$

Thus

$$2x^3 - 3x^2 - 12x + 13 = (x - 1)(2x^2 - x - 13),$$

and the solutions of Equation (8) are 1 and the solutions of

$$2x^2 - x - 13 = 0.$$

By the quadratic formula, $(1 + \sqrt{105})/4$ and $(1 - \sqrt{105})/4$ are the additional solutions of Equation (8).

Example 5. Find all zeros of

$$f\colon x \longrightarrow 12x^3 - 8x^2 - 21x + 14.$$

Solution. This is the same function that we considered earlier in Section 2-5, Example 1. At that time we found that there are zeros between 0 and 1, between 1 and 2, and between -2 and -1. Thus, we know that there are three real zeros, but we do not know whether they are rational or irrational. If all three are irrational, the best we can do is to find decimal approximations (see Section 2-7). But if at least one zero is rational, then we can obtain a function of reduced degree -- in this case a quadratic -- that will enable us to find the exact values of the remaining zeros whether rational or irrational.

If the function has a rational zero, it will be of the form p/q, and by the Rational Root Theorem the possibilities for p are ± 1, ± 2, ± 7, ± 14, and for q are 1, 2, 3, 4, 6, 12. Thus, there appear to be a good many values of p/q to test as possible zeros of the given function. But since we already know something about the location of the zeros, we need test only those possible rational zeros p/q between 0 and 1, between 1 and 2, and between -2 and -1, until a zero is found.

Now the possible rational zeros between 0 and 1 are

$$\frac{p}{q} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{2}{3}, \frac{7}{12}.$$

By synthetic substitution, we find that $f(1/2) = 3$. Since $f(0) = 14$ and $f(1) = -3$ (see Table 2-4), the zero lies between 1/2 and 1.

Hence, we need not test the values 1/3, 1/4, 1/6, and 1/12. This is a good example of how the Location Theorem may save us unnecessary work.

Continuing, we know that the only possible rational zero between 1/2 and 1 is 2/3 or 7/12. Testing these, we find that $f(2/3) = 0$, and we have found the rational zero 2/3. By the Factor Theorem, $x - 2/3$ is a divisor of $f(x)$, and the quotient, obtained from the synthetic substitution of 2/3, is

$$q(x) = 12x^2 - 21.$$

The zeros of q are the roots of

$$12x^2 - 21 = 0,$$

which are $\frac{\sqrt{7}}{4}$ and $-\frac{\sqrt{7}}{4}$.

Thus, the zeros of the given polynomial are $\frac{2}{3}$, $\frac{\sqrt{7}}{4}$, $-\frac{\sqrt{7}}{4}$.

Exercises 2-6

Find all rational zeros of the polynomial functions in Exercises 1 - 12, and find as many irrational zeros as you can.

1. a) $x \longrightarrow 2x^2 - 3x - 2$
   b) $x \longrightarrow 2x^3 - 3x^2 - 2x$
2. a) $x \longrightarrow x^3 - 6x^2 + 11x - 6$
   b) $x \longrightarrow x^4 - 6x^3 + 11x^2 - 6x$
3. a) $x \longrightarrow x^3 - 2x^2 + 3x - 4$
   b) $x \longrightarrow x^4 - 2x^3 + 3x^2 - 4x$
4. a) $x \longrightarrow 2x^3 - x^2 - 2x + 1$
   b) $x \longrightarrow 2x^4 - x^3 - 2x^2 + x$
5. $x \longrightarrow 12x^3 - 40x^2 + 19x + 21$
6. $x \longrightarrow 3x^3 - 10x^2 + 5x + 4$
7. $x \longrightarrow 4x^3 - 10x^2 + 5x + 6$
8. $x \longrightarrow x^4 - 2x^3 - 7x^2 + 8x + 12$
9. $x \longrightarrow x^4 - 8x^2 + 16$
10. $x \longrightarrow x^4 - 5x^3 + 5x^2 + 5x - 6$
11. $x \longrightarrow x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$
12. $x \longrightarrow 3x^4 - 8x^3 - 28x^2 + 64x - 15$

13. Show algebraically that the equation $x + 1/x = n$ has no real solution if n is a real number such that $|n| < 2$. (See Figure 2-5b for the geometrical picture.)
14. Find a cubic equation whose roots are -2, 1, and 3. (Hint: use the Factor Theorem.)

You are familiar with the fact that for the general quadratic equation, $ax^2 + bx + c = 0$, the sum of the roots is $-b/a$ and the product of the roots is c/a. Similar relationships exist between the roots and the coefficients of polynomials of higher degree. The following problems are intended to illustrate these relationships for third-degree polynomials.

15. Use the roots of the equation given in Exercise 14 for each of the following parts:
    a) Find the sum of the roots. Compare this result with the coefficient of x^2 obtained in Exercise 14.
    b) Find the sum of all possible two-factor products of the roots. That is, find $(-2)(1) + (-2)(3) + (1)(3)$. Compare this result with the coefficient of x obtained in Exercise 14.
    c) Find the product of the roots. Compare this result with the constant term obtained in Exercise 14.
16. If the roots of a 3rd-degree polynomial equation are -2, $1/2$, and 3, find
    a) the sum of the roots,
    b) the sum of all possible two-factor products of the roots,
    c) the product of the roots.
    d) Using the results of (a), (b), and (c), write a polynomial equation of 3rd degree having the given roots.
    e) Check your results by using the Factor Theorem to obtain the equation.
17. a) Using the Factor Theorem, write in expanded form a 3rd-degree polynomial equation having the roots r_1, r_2, and r_3.
    b) From the result obtained in part (a), and from the fact that any polynomial of 3rd degree can be written in the

form

$$a_3(x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3}),$$

find expressions for the coefficients a_2/a_3, a_1/a_3, and a_0/a_3 in terms of the roots r_1, r_2, and r_3.

18. Find the polynomial function f of degree 3 that vanishes (i.e., has zeros) at $x = -1$, 1, and 4, and satisfies the condition $f(0) = 12$.

2-7. Decimal Approximations of Irrational Zeros.

Now that we have methods for finding the rational zeros of polynomial functions, we shall discuss briefly one method for approximating a real, but irrational, zero to any number of decimal places. This may be important when there are no rational zeros, thus making it impossible by our methods to obtain a polynomial of reduced degree. For example, the polynomial

$$f: x \longrightarrow x^3 + 3x - 1$$

has no rational zeros, as you can easily verify by testing the only possibilities, 1 and -1. However, since $f(0) = -1$ and $f(1) = 3$, the Location Theorem tells us that there is a real zero between 0 and 1. Further, since $f(0.3) = -0.073$ and $f(0.4) = 0.264$, we know that the zero lies between 0.3 and 0.4.

If we interpolate between these two values, we obtain 0.32 as a better approximation of the zero. By comparing $f(0.32)$ with $f(0.31)$ and $f(0.33)$, and using the Location Theorem again, we can be certain of the value of the zero to two decimal places.

This process can be repeated indefinitely, but many people find that it isn't fun to do the arithmetic without the help of a desk calculator, and there are more powerful methods, as we shall see in Chapter 3.

Exercises 2-7

1. Find correct to the nearest 0.5, the real zero of $f: x \longrightarrow x^3 - 3x^2 - 2x + 5$ that lies between 3 and 4.

2. a) Find, correct to the nearest 0.5, the real zeros of $f: x \longrightarrow x^3 - 2x^2 + x - 3$.
   b) Find the zeros correct to the nearest 0.1.
3. a) Find a solution of $x^3 + x = 3$ correct to one decimal place.
   b) Find this solution correct to two decimal places.
4. Find the real cube root of 20 correct to two decimal places by solving the equation $x^3 = 20$.

2-8. Number of Zeros

As a result of your work with polynomials thus far, you may have the impression that every polynomial function of degree $n > 0$ has exactly n zeros. This is not quite right; what we must say is that every such function has at most n zeros. We shall prove a theorem to this effect, but first let us exhibit a polynomial function for which the number of zeros is less than the degree. The quadratic function

$$f: x \longrightarrow x^2 - 6x + 9 = (x - 3)^2$$

has only one zero, namely 3. But since the quadratic has two identical factors $x - 3$, we say that the zero 3 has multiplicity two.

We define the multiplicity of a zero r of a polynomial f to be the exponent of the highest power of $x - r$ that divides $f(x)$. That is, if

$$f(x) = (x - r)^k q(x), \quad k > 0,$$

where $q(x)$ is a polynomial, and if $x - r$ does not divide $q(x)$, then r is a zero of f of multiplicity k.

The proof of the general theorem about the number of zeros of a polynomial function depends on the fact that every such function has at least one zero. This fact, often referred to as Gauss's Theorem, is stated as follows:

Theorem 2-5. The Fundamental Theorem of Algebra. Every polynomial function of degree greater than zero has at least one zero, real or imaginary.

This is the simplest form of the Fundamental Theorem of Algebra. (As a matter of fact, the theorem is correct even if some or all of the coefficients of the polynomial are imaginary.)

The first known proof of the theorem was published by the great German mathematician Carl Friedrich Gauss (1777 - 1855) in 1799. (Eric Temple Bell has written an interesting account of Gauss. See World of Mathematics, Simon and Schuster, 1956, Volume 1, pages 295-339, or E. T. Bell, Men of Mathematics, Simon and Schuster, 1937, pages 218-269.) The proof was contained in Gauss's doctoral dissertation, published when he was 22. A translation of his second proof (1816) is in A Source Book in Mathematics, by David Eugene Smith, McGraw-Hill Book Co., 1929, pages 292-310. Gauss gave a total of four different proofs of the theorem, the last in 1850. None of the proofs is sufficiently elementary to be given here. If you study advanced mathematics in college, you may learn several proofs. You might now like to read the proof in Birkhoff and MacLane, A Survey of Modern Algebra, Macmillan, 1953, pages 107-109. Don't worry if you do not understand all of it. You may still enjoy seeing what the main idea of the proof is. (A proof is also given in L. E. Dickson, New First Course in the Theory of Equations, John Wiley and Sons, 1939.)

We are now ready to state and prove the general theorem.

Theorem 2-6. The General Form of the Fundamental Theorem of Algebra. Let f be a polynomial function of degree $n > 0$. Then f has at least one and at most n complex zeros, and the sum of the multiplicities of the zeros is exactly n.

Proof. By Theorem 2-5, f has at least one zero, say r_1. Then (recall the Factor Theorem) there is a polynomial $q(x)$ of degree $n - 1$ such that

$$f(x) = (x - r_1)\, q(x). \tag{1}$$

If $n = 1$, q is of degree zero and we have finished. If $n > 1$, the degree of q is $n - 1$ and is positive. Then, by Theorem 2-5 again, q has at least one zero r_2 (it could happen that $r_2 = r_1$) and

$$q(x) = (x - r_2)\, s(x), \tag{2}$$

where s is of degree $n - 2$. Combining (1) and (2) gives

$$f(x) = (x - r_1)(x - r_2)\, s(x). \tag{3}$$

If $n = 2$, then s in Equation (3) is of degree zero and we have finished. Otherwise, the process may be continued until we arrive at the final stage,

$$f(x) = (x - r_1)(x - r_2) \dots (x - r_n)\, z(x), \tag{4}$$

where the degree of z is $n - n = 0$. Hence, $z(x)$ is a constant. Comparison of the expanded form of Equation (4) with the equivalent form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

shows that $z(x) = a_n \neq 0$. Hence,

$$f(x) = a_n(x - r_1)(x - r_2) \dots (x - r_n). \tag{5}$$

Now, if we substitute any complex number r different from $r_1, r_2, \dots, r_n$ in place of x in Equation (5), we get

$$f(r) = a_n(r - r_1)(r - r_2) \dots (r - r_n).$$

Since every factor is different from zero, the product cannot be zero. Hence, no number except $r_1, r_2, \dots, r_n$ is a zero of f, and f has at most n zeros.

Since it is possible that some of the r_i's may be equal, the number of zeros of f may be less than n. But Equation (5) shows that f has exactly n factors of the form $x - r_i$, and therefore the sum of the multiplicities of the zeros must be n. q.e.d.

Example 1.

$$f: x \longrightarrow x^5 + x^4 - 5x^3 - x^2 + 8x - 4$$

has zeros of multiplicity greater than one. Find the zeros and indicate the multiplicity of each.

Solution. Since the coefficient of the term of highest degree is 1, we know that any rational zeros of f must be integers that are factors of 4. (Refer to Corollary 2-4-1.) Using synthetic substitution and the polynomial of reduced degree obtained each time a zero is found, we discover that 1 is a zero of multiplicity three and -2 is a zero of multiplicity two. Note that the sum of the multiplicities is five, which is also the degree of the given polynomial.

It may be helpful to show a practical way for putting down the synthetic substitutions by which we obtained the zeros and their multiplicities. This is done in Table 2-7.

Table 2-7

Finding the Zeros of f: $x \longrightarrow x^5 + x^4 - 5x^3 - x^2 + 8x - 4$

1	1	-5	-1	8	-4	1
	1	2	-3	-4	4	
1	2	-3	-4	4	0	
1	2	-3	-4	4		1
	1	3	0	-4		
1	3	0	-4	0		
1	3	0	-4			1
	1	4	4			
1	4	4	0			

1 is a zero of f of multiplicity three.

The entries 1, 2, -3, -4, 4 in the third row, 1, 3, 0, -4 in the sixth row, and 1, 4, 4 in the last row are coefficients of polynomials of degree four, three, and two, respectively. The quadratic function $x \longrightarrow x^2 + 4x + 4$ has -2 as a zero of multiplicity two since $x^2 + 4x + 4 = (x + 2)^2$.

Thus, the zeros of f are 1 (of multiplicity three) and -2 (of multiplicity two).

The graph of f is shown in Figure 2-8 in order to give you some idea of its shape in the neighborhood of the zeros -2 and 1 (points A and B). To draw this graph at the present time requires an extended table of synthetic substitutions, but in Chapter 3 methods will be developed that make it easier to determine the behavior of the graph in the vicinity of points A, B, and C.

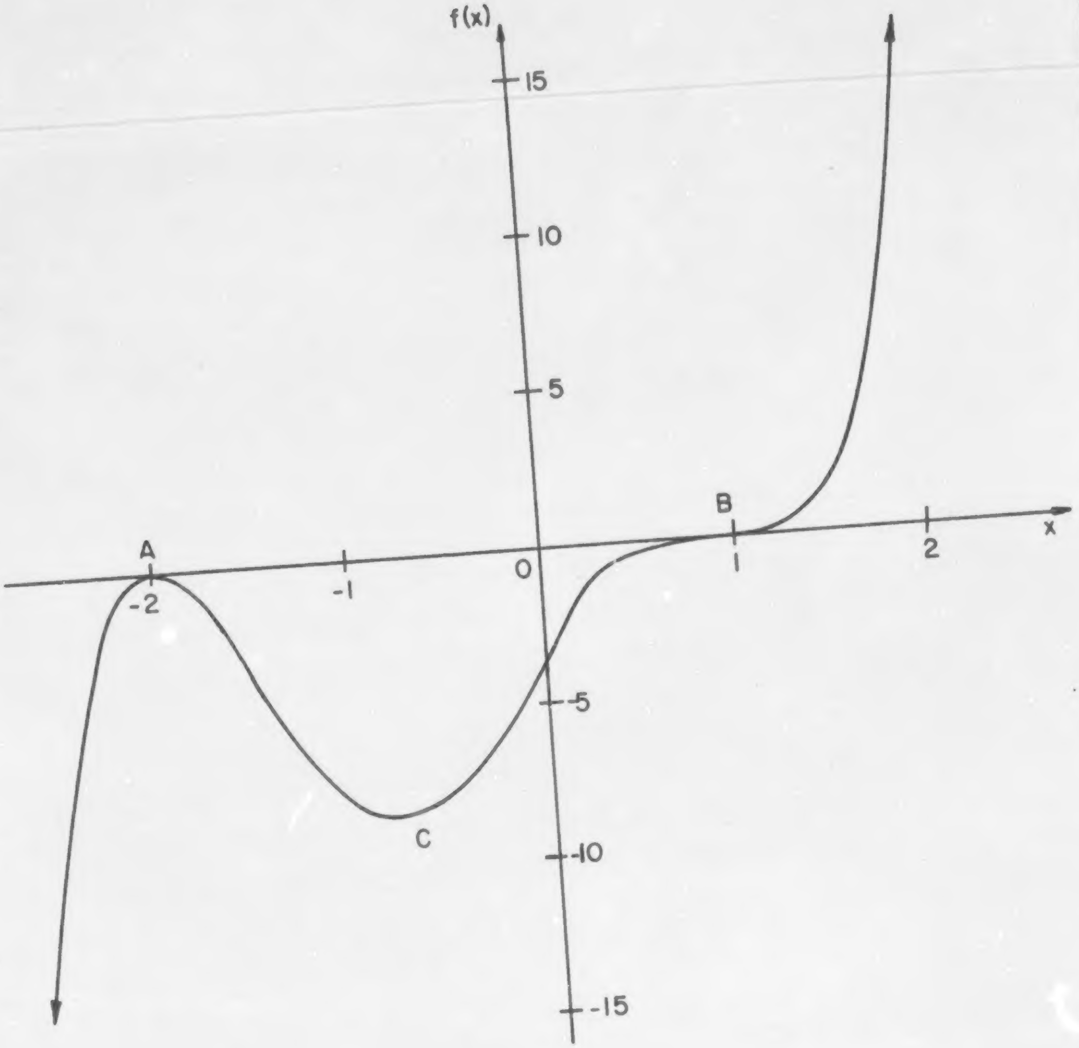


Figure 2-8.
Graph of $f: x \longrightarrow x^5 + x^4 - 5x^3 - x^2 + 8x - 4$.

The Fundamental Theorem of Algebra implies that the range of any nonconstant polynomial function includes zero when its domain is the set of all complex numbers. The range does not always include zero when the domain is the set of real numbers. For example, if

$$f: x \longrightarrow y = x^2 + 1, \quad x \in R,$$

then the range of f is the set

$$\{y : y \geq 1\}.$$

When the domain of f is the set of complex numbers, and the degree of f is > 0, then its range is also the set of all complex numbers. For, suppose that f is a polynomial of degree $n > 0$ and $a + ib$ is any complex number. Then the equation

$$f(x) = a + ib$$

is equivalent to

$$f(x) - a - ib = 0. \tag{6}$$

This is a polynomial equation of degree n; hence, by the Fundamental Theorem of Algebra, Equation (6) has a solution. That is, there exists at least one complex number x that is mapped by f into $a + ib$:

$$f(x) = a + ib.$$

Moreover, there may be as many as n different numbers in the domain that map into $a + ib$, and the sum of the multiplicities of the solutions of (6) will be exactly n.

The Fundamental Theorem does not tell us how to find even one of the zeros of f. It just guarantees that they exist. The general problem of finding a complex zero of an arbitrary polynomial is quite difficult. In the 1930's the Bell Telephone Laboratories built a machine, the Isograph, for solving such problems when the degree is 10 or less. See The Isograph -- A Mechanical Root-Finder, by R. L. Dietzold, Bell Labs Record 16, December, 1937, page 130. Nowadays, electronic computers are used to do this job, and many others. Numerous applications of computers in science and industry are discussed in a series of articles in the book The Computing Laboratory in the University, University of Wisconsin Press, Madison, Wisconsin, 1957, edited by Preston C. Hammer.

The following quotation is taken from a recent book called

Mathematics and Computers, by George R. Stibitz and Jules A. Larrivee, McGraw-Hill Book Co., New York, 1957, page 37:

"There is an interesting use for the roots of the 'characteristic equation' of a vibrating system in the dynamics of electromagnetic and mechanical systems where many of the properties of amplifiers, filters, servos, airfoils, and other devices must be determined. If any one of the complex roots of this characteristic equation for a system has a positive real part, the system will be unstable: amplifiers will howl, servos will oscillate uncontrollably, and bridges will collapse under the stresses exerted by the winds. The prediction of such behavior is of great importance to designers of the amplifiers that boost your voice as it crosses the country over telephone lines, and the servos that point guns at an attacking plane."

Exercises 2-8

1. Assume that the equations given below are the characteristic equations of some mechanical or electrical system. According to the quotation from Stibitz and Larrivee, are the systems stable or unstable?
   a) $x^3 - x^2 + 2 = 0,$
   b) $x^3 - 3x^2 + 4x - 2 = 0,$
   c) $x^3 + 3x^2 + 4x + 2 = 0,$
   d) $x^3 + x^2 - 2 = 0,$
   e) $x^3 + 6x^2 + 13x + 10 = 0.$
2. The following equations have multiple roots. Find them and, in each case, show that the sum of the multiplicities of the roots equals the degree of the polynomial.
   a) $x^3 - 3x - 2 = 0,$
   b) $x^3 - 3x + 2 = 0,$
   c) $x^4 + 5x^3 + 9x^2 + 7x + 2 = 0.$
3. Find the roots and their multiplicities of each of the following equations. Compare the solution sets of the two equations.
   a) $x^5 + 4x^4 + x^3 - 10x^2 - 4x + 8 = 0$
   b) $x^5 + x^4 - 5x^3 - x^2 + 8x - 4 = 0$

4. A number system is said to be algebraically closed if, and only if, every polynomial equation of degree > 0, with coefficients in that system, has a solution in that system. Which of the following number systems are, and which are not, algebraically closed? Give reasons for your answers.
   a) The integers: $\ldots, -2, -1, 0, 1, 2, 3, \ldots$
   b) The rational numbers.
   c) The real numbers.
   d) The pure imaginary numbers bi.
   e) The complex numbers.
5. You may have heard that it was necessary for mathematicians to invent $\sqrt{-1}$ and other complex numbers in order to solve some quadratic equations. Do you suppose that they needed to invent something that might be called "super-complex" numbers to express such things as $\sqrt[4]{-1}$, $\sqrt[6]{-1}$, and so on? Give reasons for your answers.

2-9. Complex Zeros

We know that a quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0, \tag{1}$$

has roots given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

The coefficients a, b, and c in (1) are here assumed to be real numbers. The quantity under the radical in (2) is called the discriminant. Its sign determines the nature of the roots of (1). The roots are

a) real and unequal if $b^2 - 4ac > 0$,
b) real and equal if $b^2 - 4ac = 0$,
c) imaginary if $b^2 - 4ac < 0$.

Example 1. What are the roots of $x^2 + x + 1 = 0$?

Solution. The roots are

$$\frac{-1 + i\sqrt{3}}{2}, \quad \frac{-1 - i\sqrt{3}}{2}.$$

We notice that these roots are complex conjugates; that is, they have the form $u + iv$ and $u - iv$, where u and v are real. In this example, $u = -1/2$ and $v = \frac{\sqrt{3}}{2}$.

Is it just a coincidence that these roots are complex conjugates? Let us look at (2), and suppose that a, b, and c are real numbers and that the discriminant is negative, say $-d^2$. Then the roots of $ax^2 + bx + c = 0$ are $-b/2a + i(d/2a)$ and $-b/2a - i(d/2a)$. These are complex conjugates. Thus, if a, b, and c are real and if the roots of (1) are imaginary, then these roots are complex conjugates. This is true of polynomials of any degree, as we shall now prove. (In the following theorem, the letters a and b represent the real and imaginary parts of a complex root of an equation of any degree, and do not refer to the coefficients in a quadratic expression.)

Theorem 2-7. Complex-conjugates Theorem. If $f(x)$ is a polynomial with real coefficients, and if $a + ib$ is a complex root of $f(x) = 0$, with imaginary part $b \neq 0$, then $a - ib$ is also a root.

(Another way of saying this is that if $f(a + ib) = 0$, with a and b real and $b \neq 0$, then $f(a - ib) = 0$.)

We shall give two proofs of this result.

First Proof. The key to this proof is the use of the quadratic polynomial that is the product of $x - (a + ib)$ and $x - (a - ib)$. We show that it divides $f(x)$. We can then conclude that $f(a - ib) = 0$, and we have completed the proof.

Thus, let

$$\begin{aligned} p(x) &= [x - (a + ib)][x - (a - ib)] \\ &= [(x - a) - ib][(x - a) + ib] \\ &= (x - a)^2 + b^2. \end{aligned} \tag{3}$$

Note that $p(x)$ is a quadratic polynomial with real coefficients. Now when a polynomial is divided by a quadratic, a remainder of degree less than 2 is obtained. Hence, if $f(x)$ is divided by $p(x)$, we get a polynomial quotient $q(x)$ and a remainder $r(x) = hx + k$, possibly of degree 1 (but no greater), where h, k, and all the coefficients of $q(x)$ are real. Thus,

$$f(x) = p(x) \cdot q(x) + hx + k. \tag{4}$$

This is an identity in x. By hypothesis, $f(a + ib) = 0$, and from Equation (3), $p(a + ib) = 0$. Therefore, if we substitute $a + ib$ for x in Equation (4), we get

$$0 = 0 + ha + ihb + k.$$

Since real and imaginary parts must both be 0, we have

$$ha + k = 0, \tag{5a}$$

and

$$hb = 0. \tag{5b}$$

Since $b \neq 0$ (by hypothesis), Equation (5b) requires that $h = 0$. Then Equation (5a) gives $k = 0$. Therefore, the remainder $hx + k$ in Equation (4) is zero, and

$$f(x) = p(x) \cdot q(x). \tag{6}$$

Since $p(a - ib) = 0$ by Equation (3), it follows from Equation (6) that

$$f(a - ib) = 0. \quad \text{q.e.d.}$$

Second Proof. Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \tag{7}$$

and suppose that $f(a + ib) = 0$. When we substitute $a + ib$ for x in Equation (7), we can expand $(a + ib)^2$, $(a + ib)^3$, and so on, by the Binomial Theorem. We can prove the complex-conjugates theorem, however, without actually carrying out all of these expansions, if we observe how the terms behave. Consider the first few powers of $a + ib$:

$$\begin{aligned} (a + ib)^1 &= a + ib, \\ (a + ib)^2 &= a^2 + 2aib + i^2b^2 \\ &= (a^2 - b^2) + i(2ab), \\ (a + ib)^3 &= a^3 + 3a^2ib + 3a(i^2b^2) + i^3b^3 \\ &= (a^3 - 3ab^2) + i(3a^2b - b^3). \end{aligned}$$

Now observe where b occurs in the above expanded forms. In the real parts, b either does not occur at all, or it occurs only to even powers. In the imaginary parts, b always occurs to odd powers. This follows from the fact that all even powers of i are real and all odd powers are imaginary. If we change the sign of b, we therefore leave the real part unchanged and change the sign of the imaginary part. Thus, if $f(a + ib) = u + iv$, then $f(a - ib) = u - iv$. But by hypothesis,

$$f(a + ib) = 0,$$

so that $\quad u + iv = 0,$

and therefore $\quad u = v = 0.$

Hence $\quad f(a - ib) = 0. \quad$ q.e.d.

Example 2. What is the degree of a polynomial function f of minimum degree if $2 + i$, 1, and $3 - 2i$ are zeros of f?

Solution. If it is not required that the coefficients of the polynomial be real, then we may take

$$\begin{aligned} f(x) &= [x - (2 + i)][x - 1][x - (3 - 2i)] \\ &= x^3 + (-6 + i)x^2 + (13 - 2i)x + (-8 + i). \end{aligned}$$

In this case, the degree of f is 3. No polynomial function of lower degree can have 3 zeros, so 3 is the answer. However, if it is required that the coefficients of $f(x)$ be real, then the answer to the question is 5. For then the conjugates of $2 + i$ and $3 - 2i$ must also be zeros of f. No polynomial function of degree less than 5 can have the 5 zeros

$$2 + i,\ 2 - i,\ 1,\ 3 - 2i,\ 3 + 2i. \tag{8}$$

But

$$[x - (2 + i)][x - (2 - i)][x - 1][x - (3 - 2i)][x - (3 + 2i)] \tag{9}$$

is a polynomial of degree 5, with real coefficients, that does have the numbers listed in (8) as its zeros.

Exercises 2-9

1. Multiply the factors in (9) above to show that the expression does have real coefficients. What is the coefficient of x^4 in your answer? What is the constant term? Compare these with the sum and the product of the zeros listed in (8).
2. Write a polynomial function of minimum degree that has $2 + 3i$ as a zero,
   a) if imaginary coefficients are allowed,
   b) if the coefficients must be real.
3. Find all roots of the following equations:
   a) $x^3 - 1 = 0$
   b) $x^3 + 1 = 0$
   c) $x^3 - x^2 + 2x = 8$
   d) $x^4 + 5x^2 + 4 = 0$
   e) $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$
   f) $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1 = 0$
   g) $x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1 = 0$

4. What is the degree of the polynomial equation of minimum degree with real coefficients having $2 + i$, $-2 + i$, $2 - i$, $3 + i$, $-3 + i$ as roots?
5. Consider the set of numbers of the form $a + b\sqrt{2}$, where a and b are rational. Then $a - b\sqrt{2}$ is called the conjugate surd of $a + b\sqrt{2}$. Prove the following theorem on conjugate surds:

   If $f(x)$ is a polynomial with rational coefficients, and if $a + b\sqrt{2}$ is a root of $f(x) = 0$, then $a - b\sqrt{2}$ is also a root. (Note that if $u + v\sqrt{2} = 0$, and u and v are rational, then $u = v = 0$. Otherwise, we could solve for $\sqrt{2} = -u/v$, the quotient of two rational number. But we know that $\sqrt{2}$ is irrational.)
6. Find a polynomial with rational coefficients and minimum degree having $3 + 2\sqrt{2}$ as a zero.
7. State and prove a theorem similar to that in Exercise 5 above for numbers of the form $a + b\sqrt{3}$. Is there a comparable theorem about roots of the form $a + b\sqrt{4}$? Give reasons for your answers.
8. Write a polynomial function of minimum degree that has -1 and $3 - 2\sqrt{3}$ as zeros, if
   a) irrational coefficients are allowed;
   b) the coefficients must be rational.
9. Find a polynomial of minimum degree with rational coefficients having $\sqrt{3} + \sqrt{2}$ as a zero.
10. What is the degree of a polynomial of minimum degree with (a) real, and (b) rational coefficients having
    (1) $i + \sqrt{2}$ as a zero?
    (2) $1 + i\sqrt{2}$ as a zero?
    (3) $\sqrt{2} + i\sqrt{3}$ as a zero?

2-10. Summary of Chapter 2.

This chapter deals with polynomial functions and develops methods for finding the zeros of such functions.

The general polynomial function of degree n, where n is a positive integer or zero, is denoted by

$$f: x \longrightarrow a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0, \quad a_n \neq 0.$$

f takes a given number c in its domain into the number $f(c) = a_n c^n + a_{n-1}c^{n-1} + \dots + a_1 c + a_0$ in its range.

Synthetic substitution is a technique for finding $f(c)$. It is frequently less laborious than direct substitution.

The graph of a polynomial function is plotted by preparing a table for $(x, f(x))$, using synthetic, or occasionally, direct substitution. The pairs of numbers $(x, f(x))$ are the points of the graph of f.

Theorem 2-1. Remainder Theorem. If $f(x)$ is a polynomial of degree $n > 0$ and if c is a number, then the remainder in the division of $f(x)$ by $x - c$ is $f(c)$. That is,

$$f(x) = (x - c)q(x) + f(c),$$

where the quotient $q(x)$ is a polynomial of degree $n - 1$.

The process of synthetic substitution gives a convenient way of obtaining $q(x)$ as well as $f(c)$, since the synthetic substitution of $x = c$ gives the same result as dividing $f(x)$ by $x - c$.

Corollary 2-1-1. The Factor Theorem. If c is a zero of a polynomial function f of degree $n > 0$, then $x - c$ is a factor of $f(x)$, and conversely.

Theorem 2-2. The Location Theorem. If f is a polynomial function and if a and b are real numbers such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of f between a and b.

Theorem 2-3. Upper Bound for the Zeros of a Polynomial Function. If a positive number a is substituted synthetically in $f(x)$, where f is a polynomial function, if all the coefficients of $q(x)$ are positive, and if $f(a)$ is also positive, then all the real zeros of f are less than a. We call a an upper bound for the zeros of f.

When a negative number a is substituted synthetically in $f(x)$, if the coefficients of $q(x)$ and the number $f(a)$ alternate in sign, then a is a lower bound for the zeros of f.

Theorem 2-4. *Rational Zeros of Polynomial Functions*. If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

has integer coefficients a_n, a_{n-1}, ..., a_0, and if f has a rational zero $p/q \neq 0$, $q > 0$, expressed in lowest terms, then p is a divisor of a_0 and q is a divisor of a_n.

The importance of this theorem is that it limits the number of values of x that need to be tested when searching for rational zeros of a given function.

Corollary 2-4-1. *Integral Zeros*. If

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

is a polynomial with integer coefficients, with the constant term $a_0 \neq 0$, and with the coefficient of x^n equal to 1, then the only possible rational zeros of f are integers that divide a_0.

It should be noted that if $a_0 = 0$, then 0 is a zero of f.

An equation of reduced degree can be obtained whenever a rational root of $f(x) = 0$ is found. The use of this reduced equation simplifies the problem of solving polynomial equations.

Decimal approximations of the irrational roots of a polynomial equation may be obtained to any desired accuracy by means of the Location Theorem, synthetic substitution, and interpolation, although the numerical computations may be tedious.

Theorem 2-5. *The Fundamental Theorem of Algebra*. Every non-constant polynomial function has at least one zero, real or imaginary.

Theorem 2-6. *General Form of the Fundamental Theorem of Algebra*. Let f be a polynomial function of degree $n > 0$. Then f has at least one and at most n complex zeros, and the sum of the multiplicities of the zeros is exactly n.

Theorem 2-7. *Complex-conjugates Theorem*. If $f(x)$ is a polynomial with real coefficients and if $a + ib$ is a complex root of $f(x) = 0$, with imaginary part $b \neq 0$, then $a - ib$ is also a root.

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2. Florian Cajori, A History of Mathematics, 2nd edition, Macmillan, 1922, pages 133-135. History of contributions made by Scipione del Ferro (1465-1526), Nicolo of Brescia (Tartaglia, The Stammerer) (1499-1557), Hieronimo Cardano (1501-1576), who published Tartaglia's solution of the cubic in his Ars Magna in 1545; and Ludovico Ferrari (1522-1565), whose solution of the general biquadratic equation was also published by Cardano in 1545.

3. David Eugene Smith, A Source Book in Mathematics, McGraw-Hill, 1929. Translations from Cardano's Ars Magna on the solution of the cubic (pages 203-206) and biquadratic (pages 207-212). Also A Memoir on Algebraic Equations, Proving the Impossibility of a Solution of the General Equation of the Fifth Degree by N. H. Abel (pages 261-266); and a letter written by Evariste Galois the night before a duel in which he lost his life at the age of 21. Galois introduced the technical term group in 1830, and applied the theory of groups "to discover the condition under which equations are solvable by radicals."

4. E. T. Bell, Men of Mathematics, Simon and Schuster, 1937, stories of the lives of Abel and Galois, pages 307-326 and 362-377.

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6. "On the Theory of Groups," World of Mathematics, Simon and Schuster, 1956, Volume 3, pages 1534-1573.

7. Oystein Ore, Cardano, the Gambling Scholar, Princeton University Press, 1953, page 60-107. This volume also contains a translation from the Latin of Cardano's book, Games of Chance.

Miscellaneous Exercises

1. If $f(x) = 4x^3 - 5x + 9$, find

a) $f(C)$ b) $f(-3)$ c) $f(1/2)$

2. If $f(x) = 2x - 3 + 3x^3$, find
   a) $f(0)$ b) $f(-2)$ c) $f(1/3)$
3. Find the quotient and remainder when
   a) $3x^4 - 5x^3 - 4x^2 + 3x - 2$ is divided by $x - 2$
   ~~b) $x^4 - 1$ is divided by $x + 1$~~
   c) $5x^3 - 11x^2 - 14x - 10$ is divided by $x - 3$
4. Find the quotient and remainder when
   a) $2x^3 - x^2 + 5x + 7$ is divided by $2x + 1$
   b) $81x^4 + 16$ is divided by $3x + 2$
5. Evaluate each of the following polynomials for $x = -2, -1, 0, 1, 2,$ and 3. From the information obtained, write each polynomial as a product of linear factors with rational coefficients:
   a) $x^3 - 6x^2 + 11x - 6$
   b) $x^3 - 3x^2 - 4x + 12$
   c) $x^4 - 2x^3 - x^2 + 2x$
6. If $f(x) = kx^3 + 3x^2 - 5x - 4$, find k such that when $f(x)$ is divided by $x - 2$, the remainder will be 22.
7. If $f(x) = 3x^4 - 2x^3 - 7x^2 + kx + 3$, find k such that $f(-1/3) = 0$.
8. Write each of the following quadratic expressions as a product of linear factors with real or imaginary coefficients:
   a) $4x^2 - 3$ c) $x^2 - 4x + 7$
   b) $7x^2 + 9$ d) $3x^2 - 5x - 1$
9. Locate the real zeros of f between consecutive integers, given that
   a) $f(x) = 3x^3 - 4x^2 - 8x + 5$
   b) $f(x) = x^3 - 2x^2 + 3x - 4$
10. Find all rational roots of each of the following equations. Where possible find any irrational roots.
    a) $2x^3 + x^2 - 6x = 0$
    b) $x^3 + 2x^2 - x - 2 = 0$
    c) $2x^3 - 2x^2 - 11x + 2 = 0$
    d) $3x^3 - 5x^2 - 8x + 2 = 0$
    e) $3x^4 - 8x^3 - 40x^2 + 48x + 45 = 0$

11. Compare the graphs of the functions

$$f: x \longrightarrow x^3 - 2x^2 - 5x + 6$$
$$g: x \longrightarrow 2f(x) = 2x^3 - 4x^2 - 10x + 12$$

12. Write a polynomial function of degree 3 whose zeros are -1, 2, 3.

13. Write a polynomial function of degree 3 having the zeros given in Exercise 12, with the added condition that

    a) $f(0) = -6$ b) $f(4) = 25$

14. Write a polynomial function of degree 3 that vanishes at $x = 2$ and 3, and that has the value 6 when $x = 0$ and 12 when $x = 1$.

15. Find the zero of

$$f: x \longrightarrow x^4 + 2x^3 + x^2 - 1$$

between 0 and 1 correct to one decimal place.

16. For each of the following functions find all zeros and their multiplicities, locate the y-intercept of the graph, and describe the behavior of the graph for large $|x|$.

    a) $f: x \longrightarrow y = (x - 2)^2$
    b) $f: x \longrightarrow y = (2 - x)^3$
    c) $f: x \longrightarrow y = 3(x - 2)^4$
    d) $f: x \longrightarrow y = -2(x - 1)^2(x + 2)$

17. Find all rational zeros and their multiplicities of the following polynomial functions:

    a) $f: x \longrightarrow x^5 + x^4 - 2x^3 - 2x^2 + x + 1$
    b) $f: x \longrightarrow 6x^4 + 25x^3 + 38x^2 + 25x + 6$

18. Solve the following equations:

    a) $x^2(x + 3) = 4$
    b) $(x + 1)(x + 2)(x + 3) = (x + 1)(x + 2)(x + 3)(x + 4)$

19. What is the minimum degree of a polynomial function with the following zeros?

    a) $2, \ -3, \ i$
    b) $4, \ 7 - i, \ -7 + i$
    c) $2 + i, \ -2 + i, \ -2 - i$

20. Find the minimum degree of each polynomial function with real coefficients having the zeros given in Exercise 19.

21. Find the polynomial $f(x)$ with rational coefficients and minimum degree so that $f(x) = 0$ has the following roots:
   a) $2,\ 3 - \sqrt{2}$ b) $1,\ \sqrt{2} + 3i$

22. If $f(x) = ix^3 - 3x^2 - 4$, find
   a) $f(1)$ d) $f(1 - i)$
   b) $f(i)$ e) $f(-1 + i)$
   c) $f(1 + i)$ f) $f(-1 - i)$

23. If f is a third-degree polynomial function such that $f(0) = -5$, $f(1) = -5$, $f(2) = 1$, and $f(-1) = -11$, find $f(x)$.

24. Find the quotient and remainder when
   a) $(a - b)x^2 + a^2(b - x) + b^2(x - a)$ is divided by $x - a$.
   b) $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$ is divided by $x - a$.

25. If f_n is a polynomial function with zeros a and b, prove by the Factor Theorem that $f_n(x)$ is divisible by $(x - a)(x - b)$.

26. Show by using the Factor Theorem that $x^n - a^n$ (n a positive integer) is divisible by $x - a$, and that when n is even it is also divisible by $x + a$.

27. There is a theorem known as Descartes' Rule of Signs that states that the number of positive roots of $f(x) = 0$ cannot exceed the number of variations in sign of the coefficients of $f(x)$. A variation in sign occurs whenever the sign of a coefficient differs from the sign of the next nonzero coefficient. Thus $x^4 - x^3 + 2x + 5$ has 2 variations in sign.

   Since the roots of $f(-x) = 0$ are the negatives of the roots of $f(x) = 0$, the number of negative roots of $f(x) = 0$ cannot exceed the number of variations in sign of the coefficients of $f(-x)$. Thus $f(x) = x^4 - x^3 + 2x + 5$ has at most 2 negative roots, since $f(-x) = x^4 + x^3 - 2x + 5$ has 2 variations in sign.

   Find the maximum number of positive and negative roots of each of the following equations:
   a) $x^3 - x^2 - 14x + 24 = 0$ d) $x^5 - 1 = 0$
   b) $x^7 - x^4 + 3 = 0$ e) $x^5 + 1 = 0$
   c) $3x^4 + x^2 - 2x - 3 = 0$ f) $x^5 = 0$

28. Prove that $\sqrt{2} + \sqrt{3}$ is irrational. Hint: form a polynomial with rational coefficients having $\sqrt{2} + \sqrt{3}$ as a root.

29. If $f(x) = 2x^3 - 3x + 5$, find
    a) $f(1 + i)$ b) $f(1 - i)$

30. If $f(2 + i) = 3 - 4i$ and $g(3 - i) = 2 + i$, find
    a) $f(2 - i)$
    b) $f(2 + i) \cdot g(3 + i)$
    c) $f(2 - i) \cdot g(3 + i)$
    (Hint: this problem depends on a property of polynomial functions and complex conjugates that was derived in the second proof of Theorem 2-5, and that is illustrated in Exercise 29.)

31. Given $f(x) = x^2 - 3x + 5$, compare $f(2 + \sqrt{3})$ and $f(2 - \sqrt{3})$. Does the same property hold for $g(x) = x^2 - \sqrt{3}\,x + 5$?

32. For what values of x are the following equations satisfied?
    a) $x^2 + x - 6 = 0$
    b) $|x|^2 + |x| - 6 = 0$
    c) $x + \sqrt{x} - 6 = 0$
    d) $[x]^2 + [x] - 6 = 0$
    Note: $[x]$ is the greatest integer in x, defined as the integer such that $x - 1 < [x] \le x$.

33. An examination question says, "Find the formula for the nth term of the sequence 2, 5, 10, 17, ..." Show that there is an unlimited number of polynomials in n giving these values for $n = 1, 2, 3, 4$. Find the polynomial of minimum degree answering the examination question.

34. If a, b, and c are the zeros of $f: x \longrightarrow x^3 + 7x^2 + 5$, find a polynomial function with zeros $a + 2$, $b + 2$, $c + 2$.

35. If a, b, and c are the zeros of $f: x \longrightarrow x^3 + 7x^2 + 5$, find a polynomial function with zeros $2a + 1$, $2b + 1$, $2c + 1$.

Chapter 3

TANGENTS TO GRAPHS OF POLYNOMIAL FUNCTIONS

3-1. Introduction.

If we select any point P on the graph of a polynomial function and draw a line through P with a ruler, it will be possible to choose the direction of the ruler so that very, very close to P the line seems to lie along the graph. When this is done, if we stay close enough to P, it will be impossible to distinguish between the line and the curve (see Figure 3-1a). We may appropriately refer to the straight line which has this property as the best linear approximation of the graph at P. The straight line is also said to touch or be tangent to the graph at P. In this chapter, we shall be concerned with the determination of the direction of the tangent line at any point of a polynomial graph. We shall, of course, do this neither experimentally nor inexactly, but precisely from the polynomial itself.

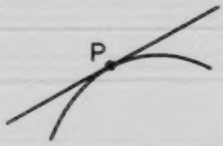


Figure 3-1a

We shall also be interested in the shape of the graph near P. That is, we shall want to know whether, sufficiently near the point P, the graph lies above or below the tangent line, or whether perhaps the curve crosses over from one side of the tangent to the other. (See Figure 3-1b. (a), (b), (c), (d).)

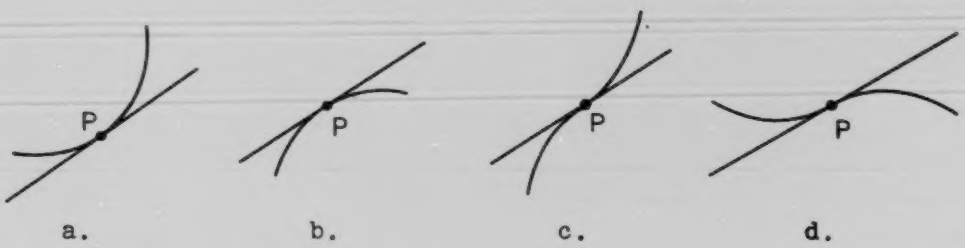


Figure 3-1b

Once we know how to determine the tangent and the shape we shall be in a position to find any points on a polynomial graph at which the tangent line is horizontal and the graph nearby is entirely above or entirely below the tangent. Such points are called minimum and maximum points, respectively. (See Figure 3-1c)

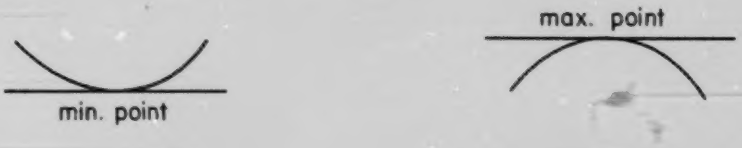


Figure 3-1c

As we learned in Section 2-3, the location of such points would help considerably in drawing the graphs and locating the zeros of polynomial functions. Maxima and minima are also of great interest in applications, as we shall see in Section 3-8.

The problem of finding the tangent to a polynomial graph at a point P and the shape of the graph nearby is particularly simple if the point is on the y-axis. As we shall see, in this case the result can be written down by inspection. At first we shall therefore confine ourselves to this easy special case, and later (Section 3-5) turn to the case in which the point is not on the y-axis.

3-2. Tangents at Points P on the y-Axis.

In this section we shall illustrate the method of obtaining an equation of the tangent to a polynomial graph at its point of intersection with the y-axis. A justification of the method will be given in Section 3-3.

The method is simplicity itself. It consists merely of *omitting every term whose degree is higher than one*.

Example 1. The graph G of $f: x \longrightarrow 1 + x - 4x^2$ intersects the y-axis at $P(0, 1)$. The tangent T to G at P has the equation

$$y = 1 + x$$

obtained by omitting the second degree term $-4x^2$. It is easy to draw T from its equation.

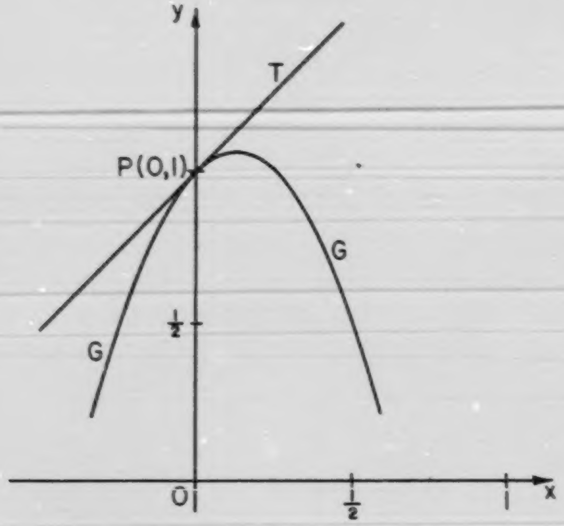


Figure 3-2a
G is the graph of f: $x \longrightarrow 1 + x - 4x^2$
T is the graph of $y = 1 + x$

Moreover, since the omitted term $-4x^2$ is negative for all values of x except 0, G lies below T except at P. (See Figure 3-2a)

Example 2. The graph G of f: $x \longrightarrow 2 + x^2$ intersects the y-axis at P(0, 2). If we omit the x^2 term and write $y = 2$ we obtain the equation of the tangent T through P. In this case the tangent is parallel to the x-axis. Since x^2 is positive for all x except zero, all points of G except P lie above the tangent line T.

Because P is the lowest point on G, it is called the minimum point of the graph. (See Figure 3-2b.)

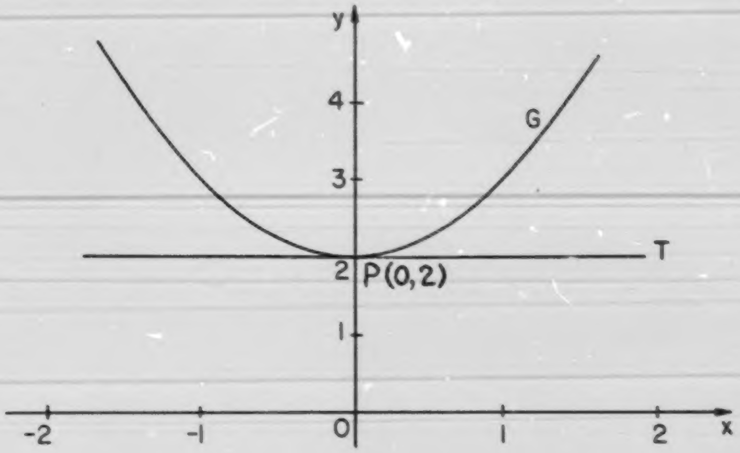


Figure 3-2b

G is the graph of f: $x \longrightarrow 2 + x^2$
T is the graph of $y = 2$

Example 3. The graph of f: $x \longrightarrow x + x^3$ intersects the y-axis at P(0, 0). The equation

$$y = x$$

of the tangent at P is obtained by omitting the x^3 term. Since x^3 is positive for positive x and negative for negative x, G is above T if $x > 0$ and below T if $x < 0$. (See Figure 3-2c.) The graph G therefore crosses from one side of the tangent to the other. P is called a point of inflection of the graph G.

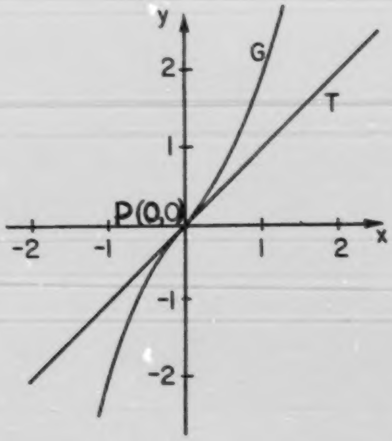


Figure 3-2c

G is the graph of f: $x \longrightarrow x + x^3$
T is the graph of $y = x$

Exercises 3-2

In each of the following Exercises find the equation of the tangent to the graph at the point P of intersection with the y-axis. Draw the tangent line and sketch the shape of the graph near P.

1.	$x \longrightarrow 1 - x + x^2$	2.	$x \longrightarrow 4 - x^2$
3.	$x \longrightarrow 2 + 3x - 2x^2$	4.	$x \longrightarrow 3 + 2x + x^4$
5.	$x \longrightarrow 1 + x + x^3$	6.	$x \longrightarrow 1 - x + x^3$
7.	$x \longrightarrow 2 - x^3$	8.	$x \longrightarrow 1 + 2x + x^5$
9.	$x \longrightarrow x + x^5$	10.	$x \longrightarrow x^4$

3-3. Why Does the Method Work? The Behavior of the Graph Near P.

The procedure of Section 3-2 is simple enough. The important question is: Why does it work? In giving the explanation it will be convenient to look at Example 1 of Section 3-2 in which

$$f\colon x \longrightarrow 1 + x - 4x^2. \qquad (1)$$

As you know we have obtained the equation

$$y = 1 + x$$

of the tangent at P (0, 1) by omitting the term $-4x^2$. We wish to justify this procedure by showing that the line obtained does represent the best linear approximation to the graph at the point P. This will entitle us to call $y = 1 + x$ the equation of the tangent to the graph at P.

From (1) we have

$$f(x) = 1 + x - 4x^2$$

which may be written as

$$f(x) = 1 + (1 - 4x)x. \qquad (2)$$

If x is numerically small, the expression $1 - 4x$ in parentheses is close to 1. In fact, we can make $1 - 4x$ lie as close to 1 as we please by making $|x|$ sufficiently small.

Specifically, if we wish $1 - 4x$ to be within .01 of 1 and hence to lie between .99 and 1.01, it will be sufficient to make $4x$ lie between -.01 and .01, and therefore to make x lie between -.0025 and .0025.

This result has a simple geometrical interpretation (see Figure 3-3a). Let us consider three lines L, L_1, and L_2 through P(0, 1) with slopes 1, 1 + .01 and 1 - .01. These lines have the equations

$$\begin{aligned} L&: \quad y = 1 + x \\ L_1&: \quad y = 1 + 1.01x \\ L_2&: \quad y = 1 + .99x \end{aligned}$$

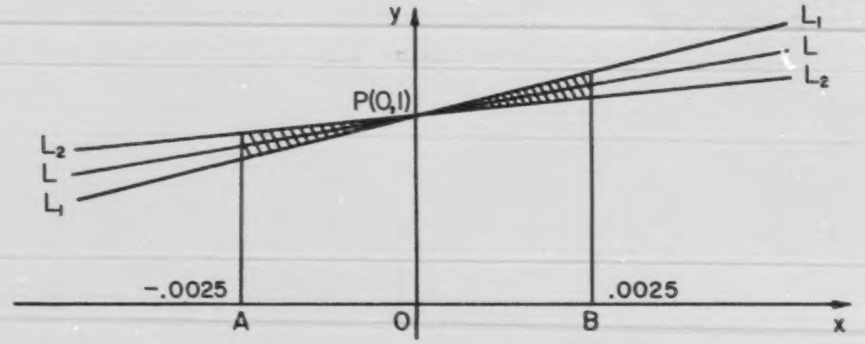


Figure 3-3a

Their slopes are so nearly equal that the differences can be shown on Figure 3-3a only by distorting the scale. Let AB be the interval $\{x: |x| < .0025\}$. If we confine ourselves to this interval AB, the graph of $f: x \longrightarrow 1 + (1 - 4x)x$ surely lies between L_1 and L_2 and, hence, in the hatched region.

The numbers chosen were merely illustrative. They were designed to give a certain concreteness to the picture. We can make $1 - 4x$ lie between $1 + \epsilon$ and $1 - \epsilon$ for an arbitrarily small value of ϵ, merely by choosing x between $-\frac{\epsilon}{4}$ and $\frac{\epsilon}{4}$. We did not need to choose $\epsilon = .01$.

This means geometrically that if we stay close enough to $x = 0$, the graph of $f: x \longrightarrow 1 + (1 - 4x)x$ lies between two lines,

$$L_1: y = 1 + (1 + \epsilon)x$$

$$L_2: y = 1 + (1 - \epsilon)x$$

which differ in direction as little as we please. The only straight line which is always included between such lines L_1 and L_2 is

$$L: y = 1 + x.$$

Hence, we see that L can indeed be regarded as the best linear approximation to $f: x \longrightarrow 1 + x - 4x^2$ at $x = 0$.

We can confine the graph G of $f: x \longrightarrow 1 + x - 4x^2$ to a smaller part of the hatched region in Figure 3-3a by noting that G lies below L except at the point P. Hence, on the interval AB, G lies between L and L_2 to the right of P and between L and L_1 to the left of P. (See Figure 3-3b)

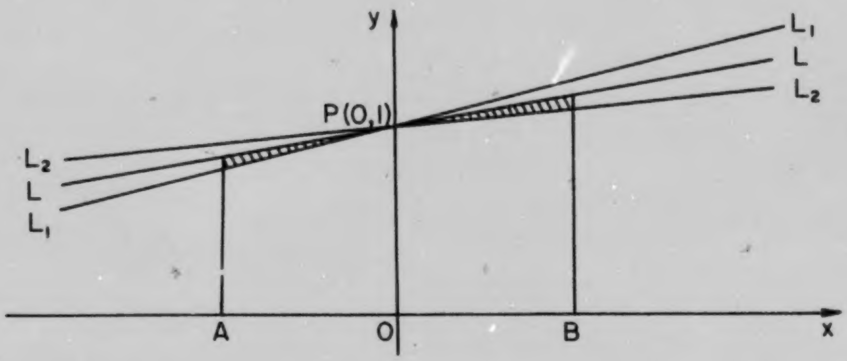


Figure 3-3b

Exercises 3-3

1. Write $f: x \longrightarrow 1 + x + x^2$ as
$$f: x \longrightarrow 1 + (1 + x)x.$$
Show that if $|x| < .01$, then $.99 < 1 + x < 1.01$ and therefore $f(x)$ lies between $1 + .99x$ and $1 + 1.01x$. Draw a figure (like Figure 3-3a) to show the geometrical meaning of this result.

2. Strengthen the result of Exercise 1 by showing that
$$1 + x < f(x) < 1 + 1.01x, \qquad \text{for } x > 0$$
$$1 + x < f(x) < 1 + .99x, \qquad \text{for } x < 0.$$
Show the improved results on a diagram.

3. Show that the results of Exercise 2 can be obtained more simply by noticing that except at P, the graph G of $f: x \longrightarrow 1 + x + x^2$ must lie above the graph of $y = 1 + x$.

4. In Example 3 write $f: x \longrightarrow x + x^3$
$$\text{as } f: x \longrightarrow (1 + x^2)x$$
Show that
   a) $x < f(x) < 1.01x, \qquad$ for $0 < x < .1$
   b) $1.01x < f(x) < x, \qquad$ for $0 > x > -.1$
   c) Draw a figure to illustrate the geometrical meaning of the results in (a) and (b).

5. Consider the function $f: x \longrightarrow 2 + 3x - x^2$
   a) At what point does the graph of the function cross the $f(x)$ axis?
   b) Show that if $|x| < .01$,
$$3.01 > 3 - x > 2.99$$
   and that $f(x)$ lies between
$$2 + 3.01x \text{ and } 2 + 2.99x.$$
   c) Draw a figure to illustrate the geometrical meaning.

6. Strengthen the result of Exercise 5 by noticing that the graph of the function lies below the graph of the straight line
$$y = 2 + 3x.$$
What additional refinement can be made in the figure associated with Exercise 5?

7. Let G be the graph of the function $f: x \longrightarrow x^2 - 2x - 1$.
   a) Write $f: x \longrightarrow x^2 - 2x - 1$ in the same form as (2) Section 3-3.

b) Show that if $0 < x < .01$, G lies between the straight lines $y = -1 - 2.01x$ and $y = -1 - 1.99x$.

c) Draw a figure to show the geometrical meaning of this result.

8. a) Write $f: x \longrightarrow 3 - 5x - 4x^2$ in the form of (2), Section 3-3.

b) If $|x| < .02$, what are the slopes of the two straight lines between which the graph of $f: x \longrightarrow 3 - 5x - 4x^2$ lies near $P\ (0, 3)$?

c) If it is desired that near $P\ (0, 3)$ the graph lies between the straight lines $y = 3 - 4.998x$ and $y = 3 - 5.002x$, what values may x assume?

3-4. The Behavior of the Graph Near P. (Continued)

In each of the examples discussed in Section 3-3 there was a single term of degree greater than one. If the polynomial contains more than one term of degree greater than one, there may be some doubt about the appearance of the graph near its intersection with the y-axis. An example will illustrate this point.

Example 1. The graph of

$$f: x \longrightarrow 1 + x + x^2 - 2x^3$$

passes through the point $P(0, 1)$. If the term $-2x^3$ had been missing, we should have no difficulty in writing the equation of the tangent T

$$y = 1 + x$$

to the graph of $x \longrightarrow 1 + x + x^2$ and concluding that near P, the graph lay above T on both sides of P.

On the other hand, if the term x^2 had been missing, we would have written $y = 1 + x$ as the equation of the tangent to the graph of $x \longrightarrow 1 + x - 2x^3$ at P and noted that P was a point of inflection with the graph below T on the right of P and above it on the left of P.

The presence of both higher degree terms raises a question. Which term dominates the situation and determines the shape? The answer is that sufficiently near $x = 0$, the lower degree term x^2 dominates the higher degree term $-2x^3$ and that the graph has the same character as if the term $-2x^3$ were missing. In fact, near P the parabola

$$x \longrightarrow 1 + x + x^2$$

gives the best quadratic approximation to the graph of

$$f\colon x \longrightarrow 1 + x + x^2 - 2x^3.$$

That this is the case may be shown by an argument like that of Section 3-3. We write

$$f(x) = 1 + x + x^2 - 2x^3$$

in the form

$$f(x) = 1 + x + (1 - 2x)x^2$$

and note that $1 - 2x$ is arbitrarily close to 1 for $|x|$ small

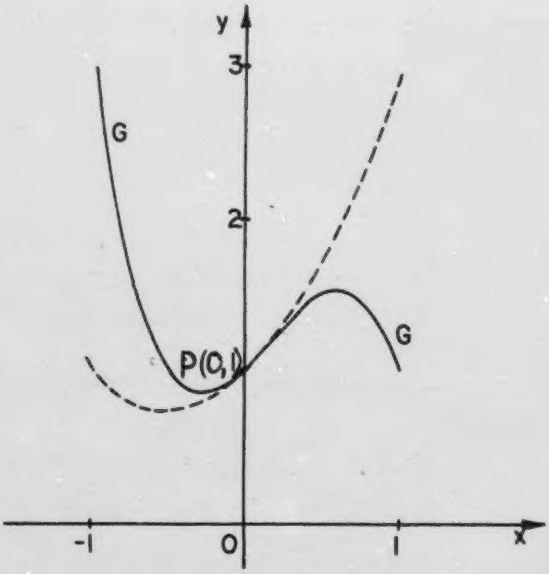


Figure 3-4a
G is the graph of $x \longrightarrow 1 + x + x^2 - 2x^3$.
The graph of $x \longrightarrow 1 + x + x^2$ is shown by the dotted line.

enough. In fact, the graph G of

$$f: x \longrightarrow 1 + x + (1 - 2x)x^2$$

lies between the parabolas

$$C_1: x \longrightarrow 1 + x + (1 + \epsilon)x^2$$

and

$$C_2: x \longrightarrow 1 + x + (1 - \epsilon)x^2$$

for arbitrarily small ϵ, provided that $|-2x| < \epsilon$, or $|x| < \frac{\epsilon}{2}$. In parabolas C_1 and C_2, the coefficients of x^2 differ from 1 as little as we please, provided, of course, that we stay close enough to $x = 0$. The only parabola which is included between all such parabolas C_1 and C_2 is $C: x \longrightarrow 1 + x + x^2$. Hence, we say that $C: x \longrightarrow 1 + x + x^2$ gives the best quadratic approximation to the graph of $f: x \longrightarrow 1 + x + x^2 - 2x^3$ for $|x|$ sufficiently near zero. The graph of $f: x \longrightarrow 1 + x + x^2 - 2x^3$ lies below C to the right of P(0, 1) and above C to the left of P(0, 1). (See Figure 3-4a.)

Example 2. Draw the graph G of $f: x \longrightarrow 2 - x + 2x^3 - 3x^4$ near its point of intersection with the f(x) - axis.

Solution: We write $f(x) = 2 - x + (2 - 3x)x^3$ and note that for $|3x| < \epsilon$, that is, for $|x| < \frac{\epsilon}{3}$, f(x) lies between

$$2 - x + (2 + \epsilon)x^3$$

and

$$2 - x + (2 - \epsilon)x^3$$

no matter how small ϵ is chosen.

We see that the required graph G lies above the line $y = 2 - x$ on the right of P and below it on the left of P, for all x sufficiently small, and hence has the same character as the graph of

$$C: x \longrightarrow 2 - x + 2x^3.$$

In fact, $2 - x + 2x^3$ is the best third degree approximation to $2 - x + 2x^3 - 3x^4$ near $x = 0$. (See Figure 3-4b.)

The conclusions drawn in these two examples would have been essentially the same if different numbers had appeared as coefficients. Thus, if

$$f: x \longrightarrow a_0 + a_1x + a_2x^2 + a_3x^3$$

we can write

$$f(x) = a_0 + a_1x + (a_2 + a_3x)x^2$$

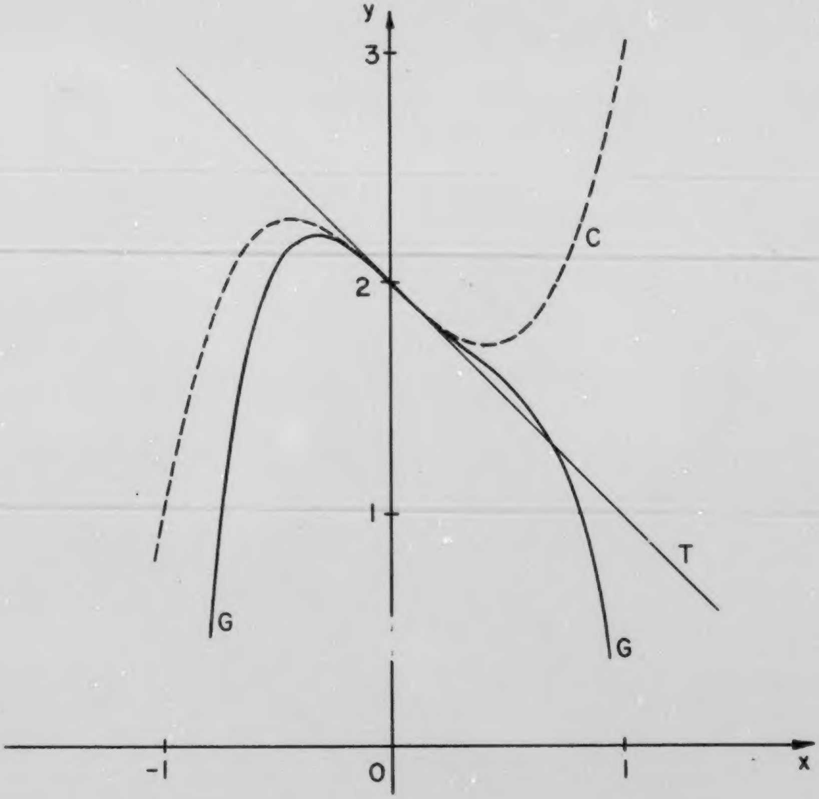


Figure 3-4b

G is the graph of $x \qquad 2 - x_3 + 2x^3 - 3x^4$

The graph of $x \qquad 2 - x + 2x^3$ is shown
by the dotted line.

and conclude that the graph lies between the graphs of

$$x \longrightarrow a_0 + a_1x + (a_2 + \epsilon)x^2$$

and

$$x \longrightarrow a_0 + a_1x + (a_2 - \epsilon)x^2$$

for arbitrarily small ϵ, provided that $|a_3x| < \epsilon$.

Exercises 3-4

For each of the following, draw the tangent and sketch the shape of the graph near its point of intersection with the f(x)-axis.

1. $f: x \longrightarrow 2 + x + 3x^2 - x^3$
2. $f: x \longrightarrow 2 + x^3 - x^4$
3. $f: x \longrightarrow -1 + 2x - x^2 + 4x^3$
4. $f: x \longrightarrow 4 - 2x^3 + x^4$
5. $f: x \longrightarrow 4 - 3x + x^3 - 7x^5$
6. $f: x \longrightarrow 2x - x^2 + 4x^3$

7 - 12. In each of the preceding exercises show that for any ϵ however small it is possible to choose $|x|$ small enough so that f(x) lies between

$$a_0 + a_1x + (a_r + \epsilon)x^r$$

and

$$a_0 + a_1x + (a_r - \epsilon)x^r,$$ where r is less than the degree of the polynomial.

Specify how small $|x|$ must be if $\epsilon = .01$.

3-5. The Tangent to the Graph at an Arbitrary Point P and the Shape of the Graph Near P.

So far we have confined ourselves to the problem of finding an equation of the tangent line to a polynomial graph at its point of intersection with the y-axis, and to an examination of the shape of the curve near that point. There remains the problem of finding the tangent to the graph at an arbitrary point P and the shape of the graph near P.

This problem is solved by generalizing the method of the previous section. The behavior near the point for which $x = 0$ was determined from the expression for f(x) in ascending powers of x. The behavior near the point for which $x = h$, say, can be determined if we have an expression for f(x) in ascending powers of $x - h$. As before, we can best start with an example.

Example 1. Find the tangent to the graph G of f: $x \longrightarrow 4 - 3x + 2x^2$ at P(1, 3).

In this case h = 1 and hence we write $f(x) = 4 - 3x + 2x^2$ in powers of x - 1. The result is

$$f(x) = 3 + 1(x - 1) + 2(x - 1)^2 \qquad (1)$$

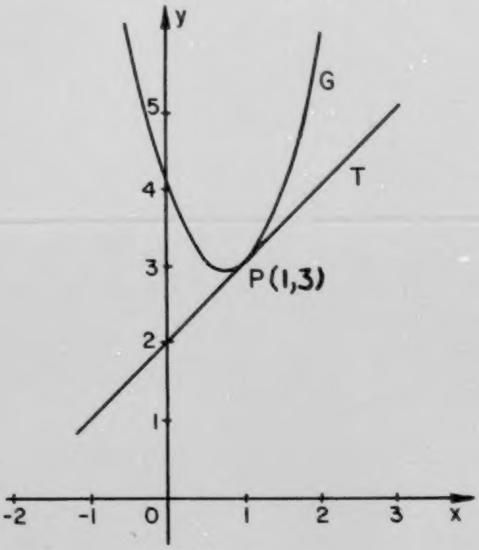


Figure 3-5

G is the graph of $x \longrightarrow 4 - 3x + 2x^2$
T is the tangent at P(1, 3)

This is easy to verify since (1) is equivalent to

$$3 + x - 1 + 2x^2 - 4x + 2 = 4 - 3x + 2x^2.$$

A method for obtaining the expansion (1) will soon be given. Meanwhile, let us see how to use (1) to achieve our purpose. We assert that the equation of the tangent T may be obtained from (1) by dropping the term of highest degree. The result is

$$y = 3 + 1(x - 1). \qquad (2)$$

The graph G is above this tangent T at all points other than P(1, 3). This is seen by noting that $3 + 1(x - 1) + 2(x - 1)^2$

may be obtained from $3 + 1(x - 1)$ by adding $2(x - 1)^2$, which is positive for all x except 1. (See Figure 3-5.)

We justify the assertion that (2) is an equation of the tangent to the curve G at point P as follows. The expansion (1) is factored and written in the form

$$f(x) = 3 + [1 + 2(x - 1)]\,(x - 1). \qquad (3)$$

From (3) we note that if x is near enough to 1, that is if $|x - 1|$ is sufficiently small, the expression $[1 + 2(x - 1)]$ is arbitrarily close to 1. In other words, for any ϵ, however small, $f(x)$ lies between

$$3 + (1 + \epsilon)(x - 1)$$

and

$$3 + (1 - \epsilon)(x - 1)$$

provided that $|2(x - 1)| < \epsilon$, that is, that $|x - 1| < \frac{\epsilon}{2}$. Hence $3 + 1(x - 1)$ is the best linear approximation to $f(x)$ near $x = 1$ and T is the tangent to the graph G at the point P(1, 3). It should be noted that we have followed the same procedures as before with $x - 1$ in place of x.

We now consider the problem of expanding $f(x)$ in powers of $x - 1$, that is, of finding the coefficients in (1). We shall discover how to do this by looking closely at (3). For convenience we repeat both (3) and (1).

$$f(x) = 3 + [1 + 2(x-1)]\,(x - 1), \qquad (3)$$

$$f(x) = 3 + 1(x - 1) + 2(x - 1)^2, \qquad (1)$$

From (3) we note that if we divide $f(x)$ by $(x - 1)$, we obtain the remainder 3 and the quotient $1 + 2(x - 1)$. The remainder 3 is the first coefficient in Equation (1). Again, from Equation (3), if we divide the quotient $1 + 2(x - 1)$ by $x - 1$, we obtain the remainder 1 and the new quotient 2. The remainder 1 is the second coefficient in Equation (1), and the final quotient 2 is the last coefficient in (1).

Let us follow this procedure to determine the required coefficients, beginning with $f(x) = 4 - 3x + 2x^2$ and using synthetic division.

$$\begin{array}{rrr}
2 & -3 & 4 \\
 & 2 & -1 \\ \hline
2 & -1 & \big|\, 3
\end{array} \qquad \underline{\big|\, 1}$$

The first remainder in dividing $f(x)$ by $x - 1$ is 3. The quotient is $2x - 1$.

On dividing $2x - 1$ by $x - 1$

$$\begin{array}{rr}
2 & -1 \\
 & 2 \\ \hline
2 & \big|\, 1
\end{array} \qquad \underline{\big|\, 1}$$

we obtain the second remainder 1 and the quotient 2. Hence, the remainders we obtained are in succession the coefficients b_0, b_1, and the final quotient is b_2.

$$f(x) = b_0 + b_1(x - 1) + b_2(x - 1)^2.$$

Example 2. Find an equation of the tangent to $f: x \longrightarrow 2 + 3x + x^2 - x^3$ at the point for which $x = 2$.

We need to expand $f(x)$ in powers of $x - 2$, that is, to find the coefficients in

$$f(x) = b_0 + b_1(x - 2) + b_2(x - 2)^2 + b_3(x - 2)^3.$$

If $f(x)$ is divided by $(x - 2)$, b_0 is the remainder and the quotient is $b_1 + b_2(x - 2) + b_3(x - 2)^2$. If this quotient is divided by $x - 2$, the remainder is b_1 and the new quotient is $b_2 + b_3(x - 2)$. A further division of $b_2 + b_3(x - 2)$ by $x - 2$ gives the remainder b_2 and the final quotient b_3. We proceed to carry out these divisions synthetically.

Dividing by $x - 2$

$$\begin{array}{rrrr}
-1 & +1 & +3 & +2 \\
 & -2 & -2 & +2 \\ \hline
-1 & -1 & 1 & \big|\, 4
\end{array} \qquad \underline{\big|\, 2}$$

we obtain the first remainder 4 and the quotient

$$-x^2 - x + 1.$$

Dividing this quotient by $x - 2$

$$\begin{array}{rrr}
-1 & -1 & +1 \\
 & -2 & -6 \\ \hline
-1 & -3 & \big|\, -5
\end{array} \qquad \underline{\big|\, 2}$$

gives the remainder -5 and the new quotient $-x - 3$. Finally, dividing this quotient by x - 2, we have

$$\begin{array}{rr|l} -1 & -3 & 2 \\ & -2 & \\ \hline -1 & \multicolumn{1}{|r}{-5} & \end{array}$$

the remainder -5 and the quotient -1. The successive coefficients in the expansion of f(x) in powers of x - 2 are the successive remainders obtained above, namely 4, -5, -5, and the final quotient -1. That is

$$f(x) = 4 - 5(x - 2) - 5(x - 2)^2 - 1(x - 2)^3.$$

The tangent T at P(2, 4) has the equation

$$y = 4 - 5(x - 2),$$

and the graph lies below T on both sides of P for points which are sufficiently near P.

Exercises 3-5

1. For each function below write the expansion of f(x) in powers of x - h and determine the equation of the tangent to the graph of f at the point $\big(h, f(h)\big)$.
   - a) $x \longrightarrow 3 + 4x + 2x^2 + x^3$ $\qquad h = 2$
   - b) $x \longrightarrow 3 + 2x^3 + 4x^2$ $\qquad h = -3$
   - c) $x \longrightarrow 4x^3 - 3x^2 + 2x + 1$ $\qquad h = -4$
   - d) $x \longrightarrow 5x^4 - 3x^2 + 2x + 1$ $\qquad h = \frac{1}{2}$
   - e) $x \longrightarrow 4x^3 + x^2 + 3x$ $\qquad h = 3$
   - f) $x \longrightarrow 2x^3 + 1x^2 - 16x - 24$ $\qquad h = -2$
2. In each case express f(x) in powers of the given factor.
   - a) $f(x) = 3x^3 - 5x^2 + 2x + 1$ $\qquad (x + 1)$
   - b) $f(x) = 2x^3 - 5x$ $\qquad (x - 2)$
   - c) $f(x) = 4 + 3x - 7x^2 + x^3$ $\qquad (x - 2)$
   - d) $f(x) = x^3 - 2x^2 + x - 1$ $\qquad (x + \frac{1}{2})$

3. For each of the following write the equation of the tangent at the specified point and sketch the shape of the graph nearby.

a)	$x \longrightarrow 4 + 3x - 7x^2 + x^3$	at (2, -10)
b)	$x \longrightarrow x^3 - 6x^2 + 6x - 1$	at (3, -10)
c)	$x \longrightarrow 3x^4 - 4x^3$	at (1, -1)
d)	$t \longrightarrow 2t^3 - 4t^2 - 5t + 9$	at (2, -1)
e)	$x \longrightarrow 2x^3 - 3x^2 - 12x + 14$	at (1, 1)
f)	$s \longrightarrow 2s^3 - 6s^2 + 6s - 1$	at (1, 1)

3-6. Application to Graphing.

Consider the function

$$f: x \longrightarrow 2 - 12x - 3x^2 + 2x^3$$

and its graph. (See Figure 3-6.)

We know how to find the tangent and sketch the graph near any point $P(h, f(h))$. So far we have chosen particular values of h. It will now be useful to carry out the work with h left unspecified. We want to expand f(x) in powers of x - h,

$$f(x) = b_0 + b_1(x - h) + b_2(x - h)^2 + b_3(x - h)^3.$$

As we know, the coefficients b_0, b_1, b_2 and b_3 can be found as the successive remainders in division by x - h.

We carry out these divisions synthetically.

$$\begin{array}{cccc|l} 2 & -3 & -12 & 2 & \underline{h} \\ & 2h & 2h^2 - 3h & 2h^3 - 3h^2 - 12h & \\ \hline 2 & 2h - 3 & 2h^2 - 3h - 12 & 2h^3 - 3h^2 - 12h + 2 & \end{array}$$

The first remainder is $f(h) = 2h^3 - 3h^2 - 12h + 2$ as we should expect. This is b_0. To obtain b_1 we divide again by x - h.

$$\begin{array}{ccc|l} 2 & 2h - 3 & 2h^2 - 3h - 12 & \underline{h} \\ & 2h & 4h^2 - 3h & \\ \hline 2 & 4h - 3 & 6h^2 - 6h - 12 = b_1 & \end{array}$$

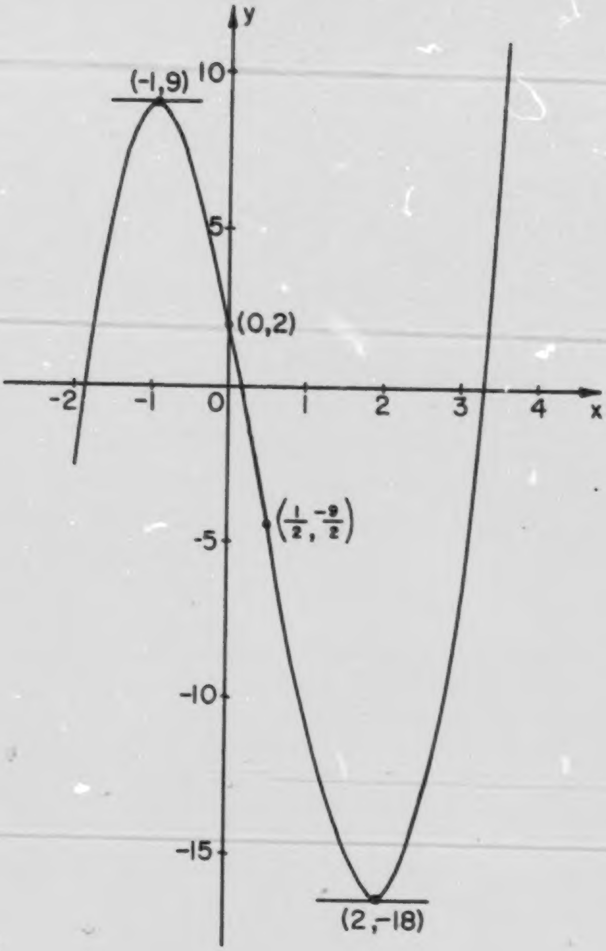


Figure 3-6
The graph of $x \longrightarrow 2 - 12x - 3x^2 + 2x^3$

To find b_2 we divide once again.

$$\begin{array}{c|ll}
2 & 4h - 3 & \underline{|h} \\
 & 2h & \\
\hline
2 & 6h - 3 = b_2 &
\end{array}$$

b_3 is 2.

The required expansion is

$$f(x) = f(h) + (6h^2 - 6h - 12)(x - h) + (6h - 3)(x - h)^2 + 2(x - h)^3 \quad (1)$$

The equation of the tangent at $\big(h, f(h)\big)$ is

$$y = f(h) + (6h^2 - 6h - 12)(x - h).$$

It is particularly helpful in graphing to find any places where the tangent is horizontal, that is, where the slope of the tangent is zero. Such points are called critical points. In our example, we set

$$6h^2 - 6h - 12 = 0$$

The solutions are $h = -1$ and $h = 2$. Since $f(-1) = 9$ and $f(2) = -18$ we have horizontal tangents at $(-1, 9)$ and $(2, -18)$, as appears on Figure 3-6.

To find the shape of the graph near $(-1, 9)$ we substitute $h = -1$ in Equation (1) and obtain

$$f(x) = 9 - 9(x + 1)^2 + 2(x + 1)^3.$$

The graph lies below the tangent $y = 9$ on both sides of $(-1, 9)$ nearby, and accordingly we call this point a relative maximum.

Similarly, if we substitute $h = 2$ in Equation (1) we obtain

$$f(x) = -18 + 9(x - 2)^2 + 2(x - 2)^3.$$

Since the graph lies above the tangent line $y = -18$ near $(2, -18)$ this point is called a relative minimum.

Another point of interest corresponds to the case where the coefficient of $(x - h)^2$ is zero. This occurs in our example when $6h - 3 = 0$, that is, when $h = \frac{1}{2}$. Equation (1) then becomes

$$f(x) = \frac{-9}{2} - \frac{27}{2}(x - \tfrac{1}{2}) + 2(x - \tfrac{1}{2})^3.$$

The tangent line T at $(\frac{1}{2}, \frac{-9}{2})$ has the equation

$$y = \frac{-9}{2} - \frac{27}{2}(x - \tfrac{1}{2}).$$

The graph lies above T to the right of the point and below T to the left. Hence, the graph crosses its tangent and $(\frac{1}{2}, \frac{-9}{2})$ is a point of inflection.

With all of the above information at our disposal we can sketch the general features of the graph of f. We use also the fact that the point (0, 2) lies on the graph and that the tangent there, $y = 2 - 12x$, has slope -12. We may find the values of the function at $x = 1$, $x = 3$ and $x = -2$ to sketch the graph more accurately. Note that in Figure 3-6 we have used different scales on the horizontal and vertical axes in order to bring out the features of the graph more clearly.

The ability to locate maximum and minimum points and points of inflection enables us to sketch the graph of a polynomial function rather quickly and it makes it possible to reduce the number of points required to give a good picture.

Exercise 3-6

For each function

$$f: x \longrightarrow 16 - 6x^2 + x^3$$

and

$$f: x \longrightarrow 2x^3 - 4x - 1$$

a) Find the slope of the tangent to the graph at the point where $x = h$.

b) Write the equation of the tangent to the graph at the point where $h = -1$.

c) Find each critical point and identify its character.

d) Evaluate f(0), f(2), f(-2), f(3), f(-3), f(10), f(-10).

e) Sketch the graph.

3-7. The Slope Function.

We can greatly shorten the process of Section 3-6. By doing synthetic division once for all on a general polynomial, we quickly discover a formula for the slope of the tangent. This formula is so easy to remember that it would be a waste of time not to use it. We illustrate this short cut by considering a general third degree polynomial function, $f: x \longrightarrow a_0 + a_1x + a_2x^2 + a_3x^3$, and finding the tangent at $(h, f(h))$.

We wish to determine the coefficients in the expansion of $f(x)$ in powers of $x - h$,

$$f(x) = b_0 + b_1(x - h) + b_2(x - h)^2 + b_3(x - h)^3.$$

In particular, we wish to find an expression for b_1, the slope of the tangent

$$y = b_0 + b_1(x - h).$$

As usual we use synthetic substitution:

$$\begin{array}{llll|l}
a_3 & a_2 & a_1 & a_0 & \underline{h} \\
 & a_3h & a_3h^2 + a_2h & a_3h^3 + a_2h^2 + a_1h & \\
\hline
a_3 & a_3h + a_2 & a_3h^2 + a_2h + a_1 & \multicolumn{1}{|l}{a_3h^3 + a_2h^2 + a_1h + a_0 = b_0} &
\end{array}$$

Another division gives

$$\begin{array}{lll|l}
a_3 & a_3h + a_2 & a_3h^2 + a_2h + a_1 & \underline{h} \\
 & a_3h & 2a_3h^2 + a_2h & \\
\hline
a_3 & 2a_3h + a_2 & \multicolumn{1}{|l}{3a_3h^2 + 2a_2h + a_1 \quad = b_1} &
\end{array}$$

The required slope is

$$b_1 = 3a_3h^2 + 2a_2h + a_1.$$

To summarize, for the polynomial function

$$f: \quad x \longrightarrow a_3x^3 + a_2x^2 + a_1x + a_0,$$

the slope at $(h, f(h))$ is

$$3a_3h^2 + 2a_2h + a_1.$$

If we associate this expression with h

$$h \longrightarrow 3a_3h^2 + 2a_2h + a_1$$

a function is defined. The same function is defined by

$$x \longrightarrow 3a_3x^2 + 2a_2x + a_1$$

since the mapping is the same no matter what letter is used to denote an arbitrary number in the domain of the function.

With the polynomial function

$$f\colon\ x \longrightarrow a_3x^3 + a_2x^2 + a_1x + a_0$$

there is therefore associated the function f'

$$f'\colon\ x \longrightarrow 3a_3x^2 + 2a_2x + a_1.$$

Since the values $f'(x)$ of this function give the slope of the tangent to the graph at the point $(x, f(x))$, we call f' the slope function associated with f.

Examination of the expression for $f'(x)$ should make it easy to remember. We can associate the terms of $f(x)$ and $f'(x)$ in accord with the following scheme:

$f(x)$	$f'(x)$
a_3x^3	$3a_3x^2$
a_2x^2	$2a_2x$
a_1x	a_1
a_0	0

In each case the degree of the corresponding term of $f'(x)$ is one lower and the coefficient is n times as great.

As we know, $f'(h)$ is the slope of the tangent to the graph of f at the point $P(h, f(h))$. We shall say that $f'(h)$ is the slope of the graph at the point P.

This scheme works equally well for polynomials of higher degree. Thus, if

$$f\colon\ x \longrightarrow a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

the slope function f' is given by

$$f'\colon\ x \longrightarrow 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1.$$

The scheme also applies to the quadratic function

$$f: x \longrightarrow a_2x^2 + a_1x + a_0$$

where

$$f': x \longrightarrow 2a_2x + a_1.$$

Example 1. Given the function $f: x \longrightarrow 1 - 2x + 3x^2$, find the associated slope function, and its value when $x = 2$.

Solution: $f': x \longrightarrow -2 + 6x$.

Hence, $f'(2) = -2 + 12 = 10$.

Example 2. Given the function $f: x \longrightarrow 3x^3 + 2x^2 - x + 1$, find the equation of the tangent line at the point $(1, 5)$.

Solution: $f': x \longrightarrow 9x^2 + 4x - 1$.

Hence, $f'(1) = 9 + 4 - 1 = 12$.

The equation of the tangent at $x = 1$ is $y = f(1) + f'(1)(x - 1)$, that is,

$$y = 5 + 12(x - 1)$$

or

$$y = 12x - 7.$$

Exercises 3-7

1. Given the functions

$$f: x \longrightarrow \frac{x^4}{4} + \frac{x^3}{3} + 2$$

$$g: x \longrightarrow 3x^5 - 5x^3 - 2$$

$$p: x \longrightarrow x^6 - 3x$$

a) Find the associated slope functions f', g', and p'.

b) Find the slope of each function at $x = -1$.

c) In each case write an equation of the tangent line at the point where the graph intersects the y-axis.

d) Sketch the graphs of f, g, and p, confining yourself to the interval $\{x: |x| < 2\}$.

2. Find and identify each critical point given the functions,
   a) $x \longrightarrow 2x^3 + 3x^2 - 12x - 7$
   b) $x \longrightarrow x^3 - 12x + 16$
   c) $x \longrightarrow -2x^3 + 3x^2 + 12x + 7$.
3. The point P(1, 1) lies on the graph of each of the following polynomial functions. For which is the point (1) a relative maximum, (2) a relative minimum, (3) a point of inflection, (4) none of these?
   a) $x \longrightarrow 2x^3 - 6x^2 + 6x - 1$
   b) $x \longrightarrow 2x^3 - 6x + 6$
   c) $x \longrightarrow 2x^3 - 9x^2 + 12x - 4$
   d) $x \longrightarrow 2x^3 - 3x^2 - 12x + 14$
4. Draw the graph of $x \longrightarrow x^4 + x^3 - 2x^2 - 3x$ after identifying each critical point, and finding the value of the function at $x = -2, 0, 2$.
5. Consider the functions

   $$f: x \longrightarrow x^3 - 3x^2 + 1 \quad \text{and} \quad g: x \longrightarrow \frac{x^2}{2} - \frac{2}{3}x - \frac{5}{6}.$$

   a) Find the associated slope functions f' and g'. Evaluate $f'(1)$ and $g'(1)$.
   b) In each case write an equation of the line tangent to the graph of the function at the point where $x = 1$.
   c) What observation can you make about the angle of intersection of these tangent lines?

3-8. Maximum and Minimum Problems.

In the last section we developed a method for finding the tangent to the graph G of a polynomial function at any point P. Moreover, in Section 3-6 we learned how to determine relative maximum and minimum values of any polynomial function of degree greater than one. We observe that a relative maximum is not necessarily greater than every other value of the function. However, in this section we shall abbreviate "relative maximum" to "maximum." If necessary, we shall refer to an over-all maximum as an absolute maximum. Similarly, we shall use the word "minimum" in place of "relative minimum."

There are many situations which lead to the problem of determining the maximum or the minimum values of polynomial functions. Such situations arise from the consideration of distance, volume, area, or cost as functions of other variables. In practice, not only do we often need to know the optimum (maximum or minimum as the situation requires) values of a function, but also how to achieve them. In other words, we need to know the values in the domain at which the function values are maxima or minima. We shall use the techniques of the last section to find these values. Sometimes the function is defined by an equation; at other times a relationship is expressed less straightforwardly and it is necessary to translate the information in such a way as to discover a function which may be maximized or minimized.

Example 1. A ball is thrown upwards so that its height t seconds later is s feet above the earth where

$$s = 96t - 16t^2.$$

What is the maximum height the ball will reach?

Solution. Our understanding of the problem is enhanced by a graph. Since meaningful replacements for t and s are limited to positive values, our graph will be drawn for the first quadrant only.

t	s
0	0
1	80
2	128
3	144
4	128
5	80
6	0

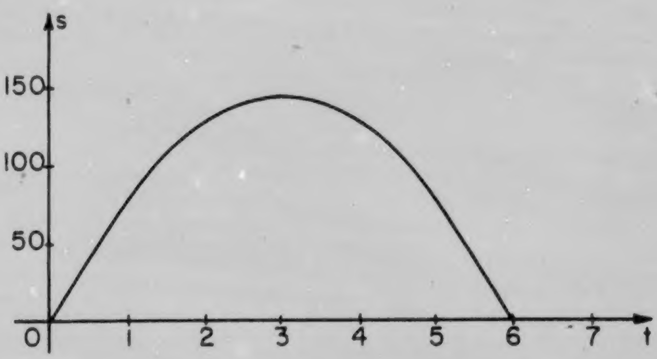


Figure 3-8a

Our graph suggests that when $t \approx 3$ ($\approx$ is read "approximately equal to"), we have a maximum height. Let us keep this in mind as we proceed.

Since the function is

$$f: t \longrightarrow 96t - 16t^2$$

the associated slope function is

$$f': t \longrightarrow 96 - 32t.$$

We are seeking a point where the tangent is horizontal, that is, where the slope is 0. Hence, we write

$$0 = 96 - 32t.$$

Thus $t = 3$, and the maximum height is 144 feet. Note that $f(t) = 96t - 16t^2 = 144 - 16(t - 3)^2$, which confirms that the value 144 is a maximum.

Example 2. Find the dimensions of the rectangle with perimeter 72 feet which will enclose the maximum area.

Solution. Let x feet represent the width and let y feet represent the length. Then the area is xy square feet. In order to express the area A in terms of x alone, we must express y in terms of x. Since the perimeter, $2x + 2y = 72$, $y = 36 - x$.

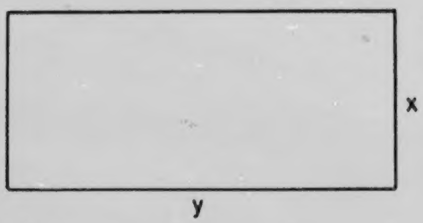


Figure 3-8b

Substituting $36 - x$ for y, we are able to express the area A in terms of x.

$$A = x(36 - x) = 36x - x^2 = f(x)$$

$$f: x \longrightarrow 36x - x^2$$

The corresponding slope function is

$$f': x \longrightarrow 36 - 2x.$$

Again we seek the values of x where $f'(x) = 0$.

$$0 = 36 - 2x$$
$$x = 18.$$

When $x = 18$, $y = 18$ since $2x + 2y = 72$. Thus the rectangle with maximum area will be a square 18 feet on a side. Why is this not a minimum?

Example 3. A man proposes to make an open box by cutting a square from each corner of a piece of cardboard 12 inches square, and then turning up the sides. Find the dimensions of each square he must cut out in order to obtain a box with maximum volume.

Solution. Let the side of the square to be cut out be x inches. The base of the box will be $12 - 2x$ inches on each side and the depth will be x inches.

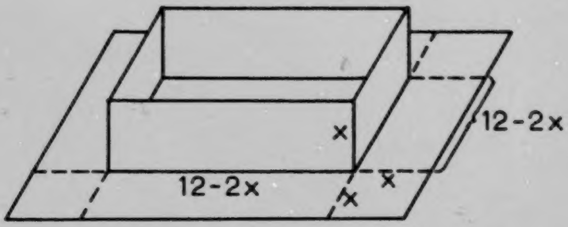


Figure 3-8c

The volume in cubic inches will be

$$V = (12 - 2x)(12 - 2x)(x)$$
$$= 144x - 48x^2 + 4x^3 = f(x).$$

We must maximize $f(x)$. The slope function is

$$f': x \longrightarrow 144 - 96x + 12x^2$$

We are seeking the zeros of the slope function. Since

$$0 = 144 - 96x + 12x^2$$
$$= 12(6 - x)(2 - x),$$

the zeros of f' are 2 and 6.

It is clear that if we cut a 6 inch square from each corner of our original cardboard, nothing will be left for a base, and the volume will be a minimum. With a 2 inch square cut from each corner we can make a box whose dimensions are 8 × 8 × 2 inches. The maximum volume is 128 cubic inches.

Example 4. Find the point on the graph of the function $f: x \longrightarrow x^2$ that is nearest the point A(3, 0).

Solution. For every real number x, the point $P(x, x^2)$ is on the graph of the given function. Recall that the distance between two points (x_1, y_1), (x_2, y_2) is given by the equation

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} .$$

A graph of the function $f: x \longrightarrow x^2$ will help. Let AP represent the distance from A to a point $P(x, x^2)$ on the graph.

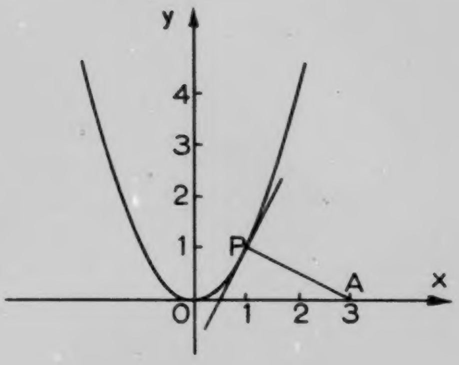


Figure 3-8d

Then

$$AP = \sqrt{(x - 3)^2 + (x^2)^2} .$$

Since this does not express a polynomial function, we do not have the technique to minimize or maximize the distance function. However, if we let$(AP)^2 = K$, then

$$K = (x - 3)^2 + (x^2)^2$$
$$= 9 - 6x + x^2 + x^4.$$

This defines a polynomial function **g** which we can minimize. The value of x which makes K a minimum will also make AP a minimum. The slope function associated with **g** is

$$g': x \longrightarrow -6 + 2x + 4x^3.$$

We find that 1 is the only real zero of **g'**. It is easy to show that when x = 1, K is a minimum and hence so is AP. Since f(1) = 1, the point (1, 1) is the point on the graph which is nearest to the point A.

Example 5. Find the right circular cylinder of greatest volume that can be inscribed in a right circular cone.

Solution. Let h be the height of the given cone and r the radius of its circular base. Then in terms of the sketch in

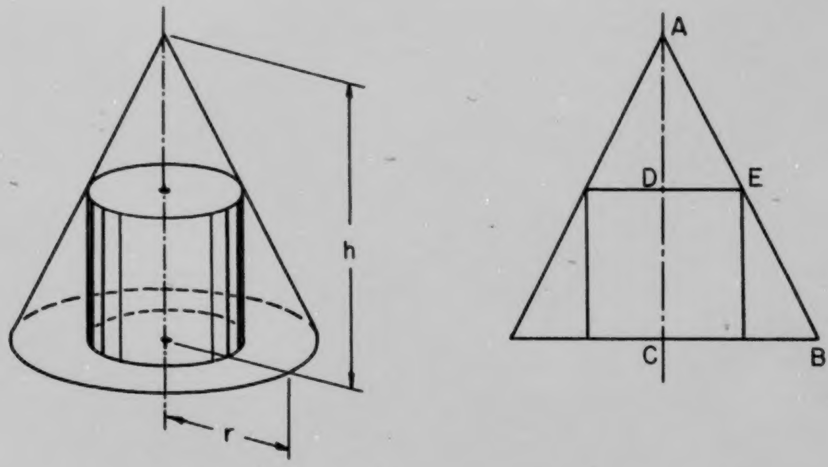


Figure 3-8e

Figure 3-8e, AC = h and CB = r. If the height of the inscribed cylinder is y, and the radius of its circular base is x, then DC = y and DE = x.

Since triangles ADE and ACB are similar

$$\frac{DE}{AD} = \frac{CB}{AC}$$

or

$$\frac{x}{h - y} = \frac{r}{h} \quad , \text{ where } AD = h - y$$

and

$$y = h - \frac{hx}{r} . \tag{1}$$

The volume V of the cylinder is given by

$$\begin{aligned} V = \pi x^2 y &= \pi x^2 \left(h - \frac{hx}{r}\right) \\ &= \pi h x^2 - \frac{\pi h}{r} x^3 , \end{aligned} \tag{2}$$

where h and r are constants.

The formula for the volume defines a polynomial function f which we can maximize. The associated slope function is

$$f': x \longrightarrow 2\pi hx - 3\pi\frac{h}{r}x^2 .$$

The zeros of f' are found by solving

$$\begin{aligned} 0 &= 2\pi hx - 3\pi\frac{h}{r}x^2 \\ &= \pi hx\left(2 - \frac{3}{r}x\right). \end{aligned}$$

These zeros are 0 and $\frac{2}{3}r$. The cylinder will have a minimum volume when the radius of its base is 0, and a maximum volume when its radius is $\frac{2}{3}r$. To find its corresponding height we substitute $x = \frac{2}{3}r$ in (1), so that

$$y = h - \frac{2}{3}h = \frac{h}{3} .$$

Exercises 3-8

1. A rectangular box with square base and open top is to be made from a 20 ft. square piece of cardboard. What is the maximum volume of such a box?
2. The sum of two positive numbers is N. Determine the numbers so that the product of one and the square of the other will be a maximum.
3. A rectangular field is to be adjacent to a river and is to have fencing on three sides, the side on the river requiring

no fencing. If 100 yards of fencing is available, find the dimensions of the field with largest area.

4. A wire 24 inches long is cut in two, and then one part is bent into the shape of a circle and the other into the shape of a square. How should it be cut if the sum of the areas is to be a minimum?

5. A man has 600 yards of fencing which he is going to use to enclose a rectangular field and then subdivide the field into two plots with a fence parallel to one side. What are the dimensions of such a field if the enclosed area is to be a maximum?

6. A printer will print 10,000 labels at a base price of $1.50 per thousand. For a larger order the base price on the entire lot is decreased by 3 cents for each thousand in excess of 10,000. For how many labels will the printer's gross income be a maximum? (See note at end of Exercises.)

7. An open box is to be made by cutting out squares from the corners of a rectangular piece of cardboard and then turning up the sides. If the piece of cardboard is 12" by 24", what are the dimensions of the box of largest volume made in this way?

8. A rectangle has two of its vertices on the x-axis and the other two above the axis on the graph of the parabola $y = 16 - x^2$. What are the dimensions of such a rectangle if its area is to be a maximum?

9. Find the point on the graph of the equation $y^2 = 4x$ which is nearest to the point $(2, 1)$.

10. A manufacturer can now ship a cargo of 100 tons at a profit of $5.00 per ton. He estimates that by waiting he can add 20 tons per week to the shipment, but that the profit on all that he ships will be reduced 25¢ per ton per week. How long will it be to his advantage to wait? (See note.)

11. Find the dimensions of the right circular cylinder of maximum volume inscribed in a sphere of radius 10 inches.

12. A peach orchard now has 30 trees per acre, and the average yield is 400 peaches per tree. For each additional tree planted per acre, the average yield is reduced by approximately 10 peaches. How many trees per acre will give the largest crop of peaches? (See note.)
13. The parcel post regulations prescribe that the sum of the length and the girth of a package must not exceed 84 inches. Find the length of the rectangular parcel with square ends which will have the largest volume and still be allowable under the parcel post regulations.
14. A rectangular sheet of galvanized metal is to be made into a trough by bending it so that the cross section has a ⊔ shape. If the metal is 10 inches wide, how deep must the trough be to carry the most water?
15. A potato grower wishes to ship as early as possible in the season in order to sell at the best price. If he ships July 1st, he can ship 6 tons at a profit of $2.00 per ton. By waiting he estimates he can add 3 tons per week to his shipment but that the profit will be reduced by 1/3 dollar per ton per week. When should he ship for a maximum profit? (See note.)
16. Prove that with a fixed perimeter P the rectangle which has a maximum area is a square.
17. Find the greatest rectangle that can be inscribed in the region bounded by $y^2 = 8x$ and $x = 4$.
18. What points on the ellipse $x^2 + 4y^2 = 8$ are nearest the point (1, 0)?
19. Find the altitude of the cone of maximum volume that can be inscribed in a sphere of radius r.
20. A real estate office handles 80 apartment units: When the rent of each unit is $60.00 per month, all units are occupied. If the rent is increased $2.00 a month, on the average one further unit remains unoccupied. Each occupied unit requires $6.00 worth of service a month (i.e. repairs and maintenance). What rent should be charged in order to obtain the most profit? (See note.)

Note. In the indicated problems meaningful replacements for the variables are obviously restricted to positive integers, but we must consider the functions to be continuous in order to apply the techniques of this chapter.

3-9. Newton's Method.

In some cases, there exist simple formulas for the zeros of functions; the linear and quadratic polynomial functions are cases in point. In some other cases, such as the cubic and quartic polynomial functions, formulas exist, but are so unwieldy as to be of little practical value. In a great many cases, there are no formulas at all. In this section, we shall consider the problem of locating zeros when no simple formula exists. We shall have to be satisfied with approximations to these zeros, but since there is no theoretical limit to the accuracy of our approximations, they can be refined to meet the demands of any practical applications.

A simple and obvious technique for approximating zeros of polynomial functions is to use the Location Theorem (Theorem 2-2) and linear interpolation. By this technique, we isolate the zero first between successive integers, then between successive tenths, then between successive hundredths, and so on, thus generating a sequence of numbers which more and more closely approximates the zero. In actual practice, this method is not generally the most efficient, and the computations may become quite involved. The method which we shall describe here is efficient and is applicable to a great variety of functions. It is called Newton's Method.

Newton's Method is an iterative process. If, that is, we start with a good approximation x_1 to the unknown zero, and apply this method, it will yield a better approximation x_2, and if we then apply it to x_2, it will yield an approximation x_3 that is better still. Thus, the process yields better and better results when applied over and over again, and if the error of the initial estimate is small, it converges more rapidly than does linear interpolation. Iterative processes are particularly well adapted to modern machine computation, and Newton's Method is a standard tool in virtually all large computing centers.

Newton's method is based on a simple geometric idea associated with successive tangents to the graph of a given function. The tangents intersect the x-axis at points which are successively closer to the zero of the function. (See Fig. 3-9a.)

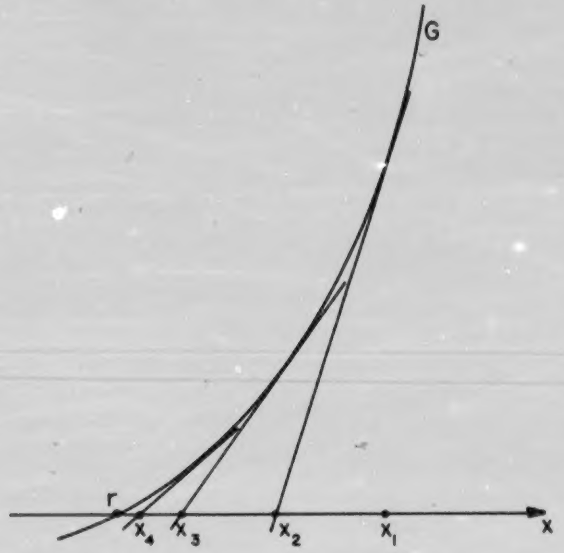


Figure 3-9a

Example 1. To calculate the value of the real zero of the polynomial function

$$f: x \longrightarrow x^3 + x^2 + x - 2$$

Solution. Since $f(0) = -2$ and $f(1) = +1$, f has a zero r which is between 0 and 1. From a graph G of f we estimate .8 as a first approximation to r. The equation of the tangent to the graph at the point where $x = .8$ is found by synthetic division.

$$\begin{array}{rrrrll}
 & & & & & \underline{|.8} \\
1 & 1 & 1 & -2 & & \\
 & .8 & 1.44 & 1.952 & & \\
\hline
1 & 1.8 & 2.44 & \multicolumn{1}{|r}{-.048} & = f(.8) & \\
 & 0.8 & 2.08 & & & \\
\cline{1-3}
1 & 2.6 & \multicolumn{1}{|r}{4.52} & = f'(.8) & &
\end{array}$$

The equation is

$$y = f(.8) + f'(.8)(x - .8)$$

or $$y = -.048 + 4.52(x - .8).$$

This tangent line crosses the x-axis near the point where $x = r$. In fact, by setting $y = 0$ we find a value of x which is a closer approximation to r.

$$\begin{aligned} 0 &= -.048 + 4.52(x - .8) \\ &= -3.664 + 4.52x \end{aligned}$$

and $$x \approx .81$$

To refine our approximation to r we find the equation of the tangent to the graph G at the point $\big(.81, f(.81)\big)$.

$$\begin{array}{rrrrll}
 & & & & & \underline{|.81} \\
1 & 1 & 1 & -2 & & \\
 & .81 & 1.4661 & 1.997541 & & \\
\hline
1 & 1.81 & 2.4661 & \multicolumn{1}{|r}{-.002459} & = f(.81) & \\
 & .81 & 2.1222 & & & \\
\cline{1-3}
1 & 2.62 & \multicolumn{1}{|r}{4.5883} & = f'(.81). & &
\end{array}$$

The equation is

$$y = f(.81) + f'(.81)(x - .81)$$

or $$y = -.00246 + 4.5883(x - .81).$$

To find the point at which this tangent line intersects the x-axis we solve the equation

$$0 = -.00246 + 4.5883(x - .81)$$

obtaining the root, $x \approx .8105$. To two decimal places the zero of the function f is .81.

Example 2. To find the real zero of

$$f\colon x \longrightarrow x^3 - 3$$

we take 1.5 as our first estimate of r since $f(1)$ and $f(2)$ have opposite signs. We wish to write the equation of the tangent to the graph G of f at the point where $x = 1.5$. In this example it is easy to evaluate $f(1.5)$ and $f'(1.5)$ by direct substitution in $f(x) = x^3 - 3$ and $f'(x) = 3x^2$. The equation is

$$y = 0.375 + 6.75(x - 1.5).$$

This tangent line intersects the x-axis at $x = 1.444$. Using 1.44 as an approximation to r we write the equation of the tangent to graph G at the point $\big(1.44, f(1.44)\big)$. The equation is

$$y = f(1.44) + f'(1.44)(x - 1.44)$$

or

$$y = -.014016 + 6.2208(x - 1.44).$$

This tangent line intersects the x-axis at $x \approx 1.442$. To two decimal places the zero of the function f is 1.44.

A common procedure is to stop the approximations as soon as two successive ones agree to the required number of places.

We now generalize the procedure in order to develop a formula for approximating an irrational zero of a function.

Let G be the graph of the given function f, and r the real zero under consideration. By inspection of the graph G, synthetic substitution, straight line interpolation, or some other device, we obtain a good one-decimal place estimate of r. Let x_1 be this approximation.

We then write the equation of the tangent to the graph G at the point $\big(x_1, f(x_1)\big)$. The equation is

$$y = f(x_1) + f'(x_1)(x - x_1).$$

This tangent intersects the x-axis at a point which we shall call $(x_2, 0)$. Under favorable conditions,

$$|x_2 - r| < |x_1 - r|,$$

and x_2 is therefore a closer approximation to r.

When the tangent crosses the x-axis we have $y = 0$ and $x = x_2$ so that

$$0 = f(x_1) + f'(x_1)(x_2 - x_1)$$

and

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = g(x_1).$$

The process may be repeated, giving successive approximations.

$$x_3 = g(x_2), \quad x_4 = g(x_3), \ \ldots$$

(See Figure 3-9a.)

It is evident that a very large amount of calculation might be needed. Ordinarily a lot of work of this sort can be justified only if it can be carried out on a calculating machine. Modern electronic computers are so fast, and so versatile, and so reliable, that it is now possible to do many large-scale calculations which were impossible, at least for practical purposes, just 15 or 20 years ago.

Computers will not do anything useful by themselves. They must be provided with a list of instruction to be followed. For comparison we can remind ourselves that a telephone exchange will do nothing useful by itself. Once the exchange is given an instruction (i.e., once the telephone number has been dialed), the circuitry reacts in such a way that a 'phone is rung and, if it is answered, the call is put through. A computer will also carry out instructions. It need not obtain its instructions one at a time, but it can follow a long list, executing automatically one instruction after another.

An important kind of instruction which can be carried out by a computer is one which causes the computer to go back in its list of instructions to an earlier one, and then repeat the intervening part of the list any number of times. Iterative calculations involve this sort of repetition of a sequence of operations. Computers are therefore well suited to the carrying out of large-scale iterations.

A list of instructions for a computer is called a program. To carry out a calculation on a computer one has to write out the program, using the proper code for the particular computer which is to be used. This program, along with any needed data, may be punched onto paper tapes. The tape is "fed" into the computer just as it is. fed into standard teletype equipment. The computer's memory stores the information while it is being used. As answers are obtained, they may for example, be printed on electric typewriters.

Exercises 3-9

1. Find $\sqrt{2}$, by Newton's method, to three decimal places. Check by a table of square roots.
2. Compute the zero of
$$f: x \longrightarrow x^3 - 12x + 1$$
between 0 and 1 to three decimal places.
3. Calculate to two decimal places the zero of
$$f: x \longrightarrow x^3 - 3x^2 + 2$$
which is between 2 and 3.
4. Find an approximate solution of
$$x^3 + 3x - 7 = 0$$
correct to two decimal places.

3-10. The Graph of Polynomial Functions Near Zeros of Multiplicity Greater Than One.

In Chapter 2 we used synthetic substitution to locate rational zeros of a polynomial function. We now inquire about the appearance of the graph near a point P on the x-axis where we have a zero r of multiplicity greater than one.

This problem is solved by extending the method of Sections 3-6 and 3-7. The behavior of the graph near the point P for which $x = r$ is determined by examining the expansion of $f(x)$ in powers of $(x - r)$. We consider a few examples.

Example 1. The polynomial function $f: x \longrightarrow x^3 - 3x - 2$ has the zero $r = -1$ of multiplicity two, since $f(x) = (x + 1)^2(x - 2)$. Using the method of Section 3-6 we expand $f(x)$ in powers of $(x + 1)$. The required expansion is

$$f(x) = 0 + 0(x + 1) - 3(x + 1)^2 + 1(x + 1)^3. \qquad (1)$$

The equation of the tangent T to the graph G at the point P(-1, 0) is

$$y = 0.$$

The graph G lies below T on both sides of P(-1, 0) for points which are sufficiently near P(-1, 0). This is seen by noting that the lower degree non-zero term $-3(x + 1)^2$ of (1) is negative for all $x \neq -1$. Hence, P(-1, 0) is a relative maximum point.

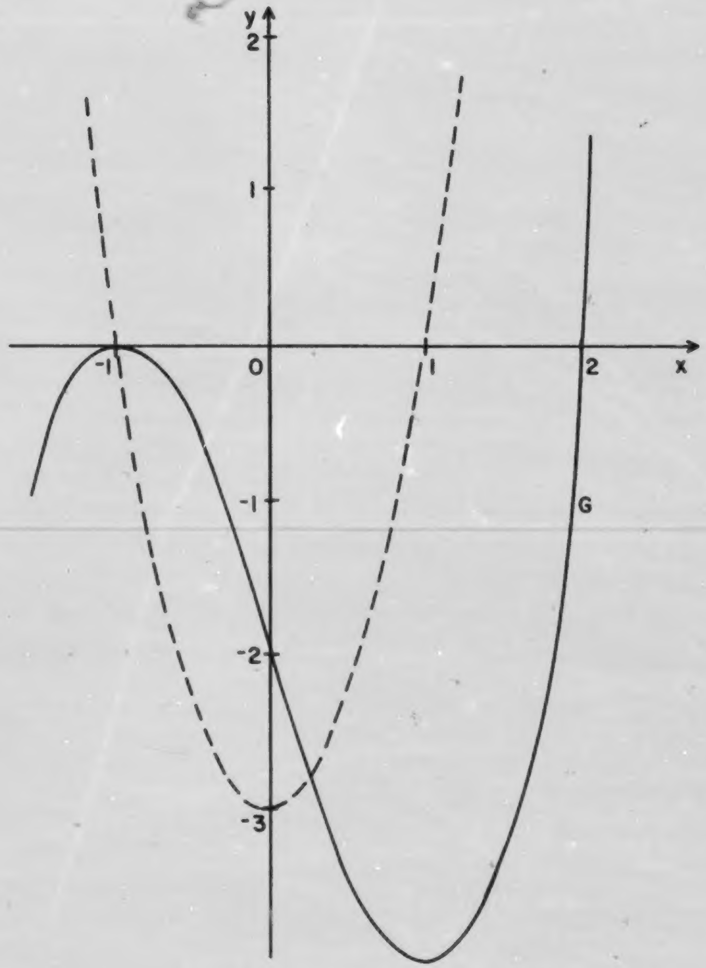


Figure 3-10a

G is the graph of $f: x \longrightarrow (x + 1)^2(x - 2)$.

Graph of $f': x \longrightarrow 3(x + 1)(x - 1)$ is indicated by the dotted line.

We know that the point (2, 0) is on the graph G since $f(2) = 0$.

In order to obtain further information about the behavior of the graph in the interval where $-1 < x < 2$ we consider the slope function

$$f': x \longrightarrow 3x^2 - 3 = 3(x + 1)(x - 1)$$

obtained by using the method of Section 3-7. We note that f' has zeros -1 and $+1$. Since we have already considered the point where $x = -1$, we now examine the point where $x = +1$ and therefore write the expansion of $f(x)$ in powers of $(x - 1)$,

$$f(x) = -4 + 3(x - 1)^2 + 1(x - 1)^3. \qquad (2)$$

This enables us to write the equation of the tangent T at the point P(1, -4). The equation is

$$y = -4.$$

From (2) we see that the graph G lies above T on both sides of P(1, -4) for all points which are sufficiently near P.

The graph of the slope function has been plotted on the same axes as graph G. (See Figure 3-10a.) Over what interval is the slope of the tangent negative? For what values of x is the slope of the tangent positive?

Example 2. To examine the behavior of the graph of $f: x \longrightarrow x^4 + 5x^3 + 9x^2 + 7x + 2$ in the vicinity of its zeros we write $f(x) = x^4 + 5x^3 + 9x^2 + 7x + 2 = (x + 1)^3(x + 2)$ and note that -1 is a zero of multiplicity three. Since

$$(x + 1)^3(x + 2) = (x + 1)^3[1 + (x + 1)] = (x + 1)^3 + (x + 1)^4$$

the expansion of $f(x)$ in powers of $(x + 1)$ is

$$f(x) = 0 + 0(x + 1) + 0(x + 1)^2 + 1(x + 1)^3 + 1(x + 1)^4. \qquad (3)$$

From the expansion we see that the x-axis (which has equation $y = 0$) is tangent to the graph G of $f: x \longrightarrow (x + 1)^3(x + 2)$ at the point P(-1, 0). The graph of G lies below the x-axis to the left of P and above the x-axis to the right of P for all points which are sufficiently near P. Thus P is a point of inflection. The point (-2, 0) is on the graph G since $f(-2) = 0$. To obtain more information about the shape of the graph G in the interval

$-2 < x < -1$ we look at the slope function

$$f': x \longrightarrow 4x^3 + 15x^2 + 18x + 7 = (x + 1)^2(4x + 7).$$

Observing that the zeros of f' are -1 and $-\frac{7}{4}$ we note that the graph has only one critical point in the interval $-2 < x < -1$. This is point $P(\frac{-7}{4}, \frac{-27}{256})$. At this point the tangent is horizontal. Since the graph G is continuous over the interval $-2 < x < -1$, it is reasonable to expect that the ordinate increases steadily from P to the critical point $(-1, 0)$. It is intuitively quite easy to see that if the graph G has horizontal tangents at A and B, but at no point between A and B, the graph rises steadily or falls steadily from A to B. (See Figure 3-10b.)

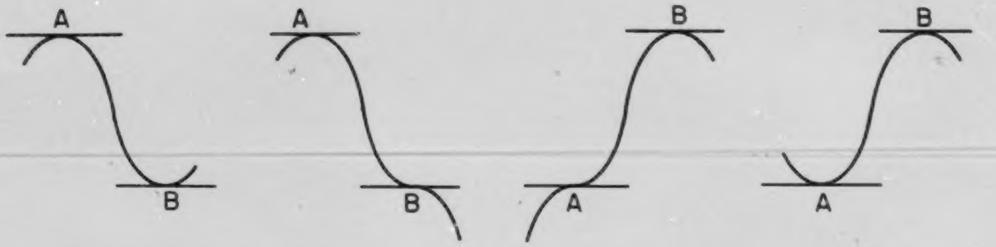


Figure 3-10b

In terms of the graph of G we note that $P(\frac{-7}{4}, \frac{-27}{256})$ is a relative minimum point. An examination of the graph of the slope function $f': x \longrightarrow (x + 1)^2(4x + 7)$ in the neighborhood of $x = \frac{-7}{4}$ shows that the slope of the tangent to G is negative for $x < \frac{-7}{4}$ and positive for $x > \frac{-7}{4}$. (See Figure 3-10c.)

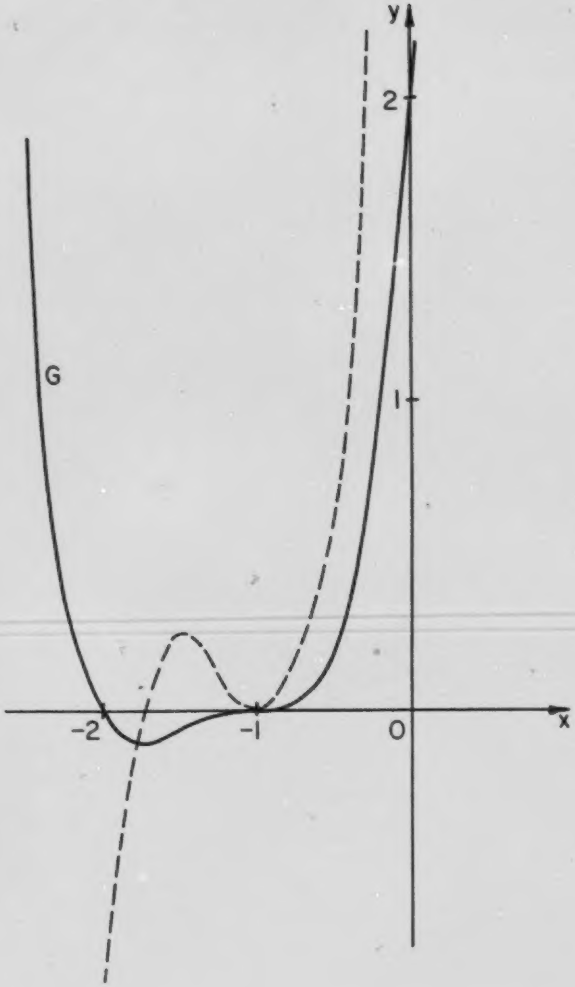


Figure 3-10c
G is the graph of f: $x \longrightarrow (x + 1)^3(x + 2)$
The dotted line shows the graph of the slope function f': $x \longrightarrow (x + 1)^2(4x + 7)$.

Exercises 3-10

1. Consider f: $x \longrightarrow x^3 - 3x + 2$
   a) Locate the zeros of the function.
   b) Locate each relative maximum, relative minimum, and point of inflection.
   c) Sketch the graph.
2. Draw the graph of f: $x \longrightarrow x^3 - 3x^2 + 4$ by finding zeros of the function and points where the slope of the tangent is zero.
3. For each function test each zero of multiplicity more than one for possible points of inflection.
   a) f: $x \longrightarrow x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$
   b) f: $x \longrightarrow x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1$
   c) f: $x \longrightarrow (x - 1)^3(x^2 + 1)$
   d) f: $x \longrightarrow (x - 1)^3(x + 1)(x^2 + 1)$

3-11. Summary of Chapter 3.

If P is the point $(h, f(h))$ on the graph G of the polynomial function f: $x \longrightarrow f(x)$, there exists a straight line T through P which is called the tangent to G at P. T is the best linear approximation to G at P in the following sense. Let m be the slope of T and ϵ an arbitrarily small positive number. Then if $|x - h|$ is small enough, all points of G lie in the region bounded above and below by the straight lines through P with slopes $m + \epsilon$ and $m - \epsilon$. There is only one slope m that has this property, and hence only one line that is tangent to G at P.

To find the slope m, we may use repeated synthetic division by $x - h$ to write f(x) in powers of $x - h$,

$$f(x) = b_0 + b_1(x - h) + b_2(x - h)^2 + \ldots + b_n(x - h)^n. \qquad (1)$$

Then $m = b_1$, the coefficient of $x - h$.

The shape of G near P is determined by the first term in (1) of degree greater than one with a coefficient different from zero. If b_2 is positive and $|x - h|$ is small enough, G lies above T on both sides of P. If b_2 is negative and $|x - h|$ is small enough, G lies below T on both sides of P. If $b_2 = 0$, we study the sign of the first subsequent term with nonzero coefficient, for $x > h$ and for $x < h$.

The slope m of T may be obtained quickly from the slope function f', where if

$$f: x \longrightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

then
$$f': x \longrightarrow a_1 + 2a_2x + \ldots + na_nx^{n-1}$$

and
$$m = f'(h).$$

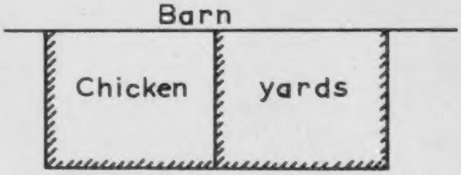
Of particular importance are points of G at which $f'(h) = 0$, so that T is horizontal. If, near such a point P, G lies above T, P is called a relative minimum; if below T, a relative maximum.

Applications of these ideas are made to the plotting of graphs (in particular to their shape near points where f has a zero of multiplicity greater than one) and to the solution of maximum-minimum problems. Also, by replacing a graph by its tangent near an intersection with the x-axis, a method (Newton's) is developed for calculating successive approximations to irrational zeros of polynomial functions.

Miscellaneous Exercises

1. Given $f: x \longrightarrow 2x^2 - 7x - 4$. If $x_2 = x_1 + 2$, how much greater is the slope at x_2 than at x_1?
2. Point $P(-3, 2)$ lies on the graph of $f: x \longrightarrow ax^2 + bx + 8$. If the slope of f at P is -1, find a and b.
3. If $(6, 0)$ and $(2, 0)$ lie on the graph of $f: x \longrightarrow ax^2 + bx + c$, and if the slope of f at $(6, 0)$ is 5, what is its slope at $(2, 0)$?
4. Write the linear approximation for $f: x \longrightarrow x^2 - 4x + 3$ at $x = 2$ and at $x = 4$. Using these two linear approximations for f, find the error at $x = 2.01$ and at $x = 4.01$. If the error committed in using the linear approximation for f at $\big(h, f(h)\big)$ is to be numerically less than 0.01, then $|x - h|$ must be less than what number?
5. If $f: x \longrightarrow 5(x - h)^2 + (x - h) + 3$, write $f(x)$ in powers of x.
6. For each function below, write the equation of the tangent to the graph at $\big(0, f(0)\big)$ and sketch the graphs of the function and the tangent in the vicinity of this point:
   a) $f: x \longrightarrow f(x) = 3x^2 - 2x + 1$
   b) $f: x \longrightarrow f(x) = 4 - x^2$
7. If $f: x \longrightarrow 2x^3 - 5x^2 + x + 3$, write $f(x)$ in powers of $(x + 2)$.
8. If $f(x) = a(x - h)^2 + b(x - h) + c$, show that $f(x) = a(x - 1)^2 + (2a + b - 2ah)(x - 1) + f(1)$.
9. Given the function $x \longrightarrow f(x) = 4 - 3x + 2x^2$, find the equation of the tangent at $\big(h, f(h)\big)$ and find the value of h for which the tangent is horizontal. Sketch the graph of the function and its horizontal tangent.
10. Prove that a quadratic function cannot have a point of inflection.
11. Express the polynomial $2x^3 - 5x^2 + 3x - 4$ in the form $a_0 + a_1(x - 3) + a_2(x - 3)^2 + a_3(x - 3)^3$.

12. A farmer plans to enclose two chicken yards next to his barn with fencing, as shown. Find
    a) the maximum area he can enclose with 120 feet of fence;
    b) the maximum area he can enclose if the dividing fence is parallel to the barn.



13. If a clay pigeon is shot vertically from the ground at a speed of 88 ft./sec., its distance d from the ground in feet is $d = 88t - 16t^2$, where t is the time in seconds. In how many seconds does the pigeon reach maximum height? What is this height? In how many seconds from the time it is shot does it hit the ground?
14. If $f(x) = 2x^3 + x^2 - 3x + 4$, what quadratic function is an approximation for f near $x = 0$? What is the difference between the slope of this quadratic and the slope of f at $x = 0$?
15. Show that $x + a$ is a factor of $x^7 + a^7$. Using synthetic substitution find the other factor.
16. Prove that if $f: x \longrightarrow ax^2 + bx + c \quad (a > 0)$ has zeros x_1 and x_2, then f has a minimum at
$$x = \frac{x_1 + x_2}{2}.$$
17. Suppose (in using Newton's method) our first guess is lucky, in the sense that x_1 is a zero of the given function. What happens to x_2 and later approximations?
18. Find the root of $x^3 - 3x + 1 = 0$ between 0 and 1 correct to 2 decimal places.
19. The graph of a function g passes through the point (2, 3). Its slope function is
$$g': x \longrightarrow 3x + 2.$$
Write an equation describing g.
20. At what point on the graph of $f: x \longrightarrow x^2 + 5x + 4$ does f have a slope of -5?

21. Give a formula for all quadratic functions f whose graphs pass through P(0, -3) and have slope 4 at P.

22. Determine the behavior of the graph of

$$f: x \longrightarrow x^3 - 3x$$

near $x = 0$. Graph the function for $|x| < 2$.

23. A rectangular pasture, with one side bounded by a straight river, is fenced on the remaining three sides. If the length of the fence is 400 yards, find the dimensions of the pasture with maximum area.

24. A box is x feet deep, $mx + b$ feet wide, and y feet long. Its volume is C cubic feet. Find a function f such that $y = f(x)$.

25. The graph of a quadratic function f has a maximum at (-1, 1) and passes through (0, 0).

    a) Graph the function and write its equation.

    b) Write the equation of the quadratic function whose graph is symmetric to f with respect to the line $y = 1$.

26. Find the quadratic polynomial function whose graph passes through the origin and which has a maximum at (2, 3).

27. For what range of values of x is the error numerically less than 0.01 when $4x^2 + 3x - 2$ is replaced by its best linear approximation at $x = 0$?

28. The graph of a third degree polynomial function has a relative maximum at (-1, 2) and a relative minimum at (1, -2). Write an equation describing the function. Find the slope of f at $x = 0$ and show that (0, 0) is a point of inflection. Compare the slope of f at $x_1 = h$ and $x_2 = -h$. Sketch the graph of f.

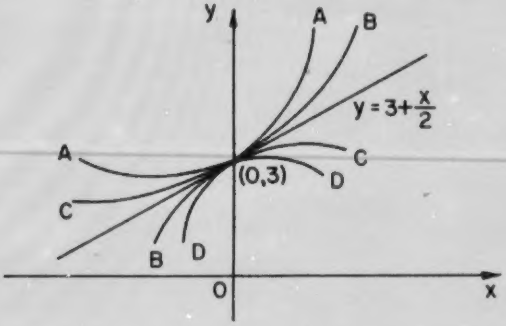
29. Write the expansion of $g(x) = x^3 - 9x^2 + 24x - 18$ in powers of $x - 3$. Find the slope of g at $x = 3$, and show that (3, 0) is a point of inflection. Compare the slope of g at $x_1 = 3 + k$ and $x_2 = 3 - k$. Sketch the graph of g.

30. Compare the graph of f in Exercise 28 with the graph of g in Exercise 29. If $g(x) = f(x + h)$ what is the value of h? Write the equation describing g_1 if $g_1(x) = f(x + 2)$, and sketch the graph of g_1.

31. Is the graph of f in Exercise 28 symmetric with respect to the y-axis or to the origin? Justify your conclusion. Answer the same question for the graph of g in Exercise 29.

32. Classify each of the points (1, 0), (2, -2) and (3, -4) on the graph of $x \longrightarrow x^3 - 6x^2 + 9x - 4$ as a relative maximum, a relative minimum, a point of inflection, or none of these.

33.



The figure at the left shows four polynomial graphs and their common tangent $y = 3 + \frac{x}{2}$ at (0, 3). Match each graph (A, B, C, D) with one of the following equations.

1) $y = 3 - \frac{x}{2} - x^3$
2) $y = 3 - \frac{x}{2} - x^2$
3) $y = 3 - \frac{x}{2} + x^2$
4) $y = 3 - \frac{x}{2} + x^3$
5) $y = 3 + \frac{x}{2} - x^3$
6) $y = 3 + \frac{x}{2} - x^2$
7) $y = 3 + \frac{x}{2} + x^2$
8) $y = 3 + \frac{x}{2} + x^3$

34. Find and classify each critical point of the following functions:

a) $x \longrightarrow (x - 2)^2$
b) $x \longrightarrow (2 - x)^3$
c) $x \longrightarrow (x - 2)^4$
d) $x \longrightarrow (x - 1)^2(x + 2)$

35. Find, correct to 3 decimals, the zero of $x \longrightarrow x^4 + 2x^3 + x^2 - 1$ which is between 0 and 1.

36. Find all real roots of $x^3 - 3x^2 + 2x - 1 = 0$ correct to 3 decimals.

37. For what set of values of k will

$$x \longrightarrow 2x^3 - 9x^2 + 12x + k \qquad \text{have}$$

a) no real zeros,

b) one real zero,

c) three real zeros.

38. If $f: x \longrightarrow a(x - h)^2 + b(x - h) + c = Ax^2 + Bx + f(k)$, express $f(k)$ in terms of a, b, c, and h.

39. A right triangle whose hypotenuse is 5 is rotated about one leg to form a right circular cone. What is the largest volume which the cone can have?

40. Using Newton's Method find the real root of

$$x^3 - 2x - 5 = 0$$

correct to the nearest 0.001.

41. Find the value of k for which the maximum point of the graph of $f: x \longrightarrow 2k + 3x - 5x^2$ has the same x and y coordinates.

42. What number exceeds its square by the greatest possible amount? Prove your result.

43. Find the point on the circle $x^2 + y^2 = 9$ which is nearest the point $(5, 0)$.

44. Find the maximum value of the function

$$x \longrightarrow \frac{2}{x^2 - 6x + 10}$$

45. Find the maximum value of the function $x \longrightarrow x^2 - 6x + 10$ with domain $\{x: 1 \leq x \leq 4\}$.

46. Write the polynomial which is the best third degree approximation to

$g(x) = 6x^5 - 4x^3 + 5x + 2$, for $|x|$ near zero.

Then find the value of $g(0.1)$ correct to three decimal places. Show that your result gives the slope of the function $f: x \longrightarrow x^6 - x^4 + 2.5x^2 + 2x - 6$ correct to the nearest 0.001.

47. Find an equation of the tangent to the graph of $f: x \longrightarrow x^3 + 3x^2 - 4x - 3$ at its point of inflection.

48. Locate and identify the character of the critical points of

$$g: x \longrightarrow 3x^4 - 12x^3 + 12x^2 - 4.$$

49. Find the zeros of $f: x \longrightarrow (x - 3)^2(x + 4)^3$. Sketch the graph near each zero.

50. Using Newton's method, compute the real root of $x^5 + 5x - 1 = 0$ correct to three decimal places.

51. Prove that if $f: x \longrightarrow a_3x^3 + a_2x^2 + a_1x + a_0$ has a relative maximum at $x = x_1$ and a relative minimum at $x = x_2$, then it has a point of inflection at

$$x = \frac{x_1 + x_2}{2}.$$

52. If $g(x) = cf(x)$, where c is a constant and f is a polynomial function, show that $g' = cf'$. Hint: Assume that $f(x) = b_0 + b_1(x - h) + b_2(x - h)^2 + \ldots + b_m(x - h)^m$ and compare $cf'(h)$ with $g'(h)$. Since this holds for any h, we have the required result.

53. If $s(x) = f(x) + g(x)$, where f and g are polynomial functions, show that $s' = f' + g'$.

54. Use the results of Exercises 52 and 53 to show that the determination of the slope function of a polynomial reduces to finding the slope function of x^k for $k = 1, 2, 3, \ldots$

55. Show that $P_1(0, 0)$ and $P_2(-1, -11)$ are points of inflection of the graph of

$$f: x \longrightarrow 2x^6 + 3x^5 + 10x.$$

56. Show that $P(\frac{5}{2}, \frac{-15}{16})$ is a relative minimum point of the graph of

$$f: x \longrightarrow x^4 - 2x^3 - 7x^2 + 10x + 10.$$

Then by the Location Theorem show that f has two real zeros between 2 and 3. Locate each of the other two real zeros between consecutive integers.

57. Sketch the graph of $f: x \longrightarrow x^3 - x^2 - 3x + 1$ by approximating the abscissas of the relative maximum and minimum points. Find the smallest root of $x^3 - x^2 - 3x + 1 = 0$ correct to two decimal places.

58. Find the slope function f' associated with

$$f: x \longrightarrow x^4 - 2x^3 + 3.$$

Locate the critical points of the slope function. Draw the graphs of f and f' on the same coordinate axes. Describe the behavior of the graph G of f near points of G which are directly above or below the critical points of f'.

59. In using Newton's method, suppose that our guess is unlucky in the sense that $f'(x_1) = 0$. What happens in this case?

60. Show that the function $x \longrightarrow |x|$ attains its minimum value at $x = 0$. What is the slope function? Why cannot the slope function be used to find the minimum point?

Chapter 4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

4-1. Introduction

In the two preceding chapters, we studied polynomial functions, tangents to their graphs, maxima and minima, and so on. In this chapter, we take up a totally new class of functions, called exponential functions. We shall also study logarithmic functions, which are related in a special way to exponential functions.

Exponential functions have appeared in mathematics through two different avenues. One of these is through ordinary powers of a positive number. Consider for example the number 2. We know very well what the symbols 2^1, 2^2, 2^3, 2^4, ... mean: $2^1 = 2$, $2^2 = 2\cdot 2 = 4$, $2^3 = 2\cdot 2\cdot 2 = 8$, $2^4 = 2\cdot 2\cdot 2\cdot 2 = 16$, and so on. We then define $2^0 = 1$. Negative integral powers of 2 are defined as reciprocals of the corresponding positive integral powers: $2^{-1} = 1/2$, $2^{-2} = 1/4$, $2^{-3} = 1/8$, $2^{-4} = 1/16$, and in general $2^{-n} = 1/2^n$, for every positive integer n.

We also know how to define rational powers of 2 for rational exponents m/n that are not integers. For example, we write $2^{1/2}$ for the positive number whose square is 2. Similarly, $2^{1/3}$ is the positive number whose cube is 2, and so on. For every positive integer n, $2^{1/n}$ is the positive number whose n-th power is 2. Extending our definition, we write $2^{3/5}$, for example, for the number $(2^{1/5})^3$. Thus, if we write r to stand for a positive rational number, we know what 2^r means.

Negative rational powers of 2 are defined as the reciprocals of the corresponding positive rational powers: $2^{-1/2} = \frac{1}{2^{1/2}}$, and in general $2^{-r} = \frac{1}{2^r}$ for every positive rational number r.

Rational powers of 2 obey two useful laws:

$$2^{r+s} = 2^r 2^s \tag{1}$$

and

$$(2^r)^s = 2^{(rs)} \tag{2}$$

for all rational numbers r and s, positive, zero, or negative.

We can think of the rational powers of 2 as defining a function $f: r \to 2^r$ whose domain is the set of all rational numbers r.

The second avenue through which exponential functions appear in mathematics is the study of various natural phenomena. For example, a biologist grows a colony of a certain kind of bacteria in a Petri dish in his laboratory. As part of his investigation he wishes to study how the number of bacteria changes with time. Under favorable circumstances it is found that so long as the food holds out, the time required for the number of bacteria to double does not depend on the time at which he starts the experiment. This is a special case of a general principle of growth which is of great importance in many sciences, social and physical, as well as biological. We shall study this special case and abstract from it certain important mathematical ideas.

To be concrete, let us suppose that on a given day there are N_0 bacteria present and that the number of bacteria doubles every day. Then there will be $2N_0$ present one day after the start of the experiment. After another day the number of bacteria will be twice $2N_0$ or 2^2N_0, after three days twice 2^2N_0 or 2^3N_0, after n days the number N of bacteria present will be given by the equation

$$N = N_0 2^n \tag{3}$$

where n is a positive integer.

If we do not assume that the number of bacteria jumps suddenly every 24 hours, but rather that N increases steadily throughout any given day, we might ask ourselves such questions as the following: How many bacteria are present 1/2 day or 1 1/4 days after the start? How many were present 2 days before the experiment started, that is, before the initial count was made?

You may perhaps guess that these questions are answered by generalizing the equation (3) to

$$N(r) = N_0 2^r \qquad (4)$$

where r is not restricted to positive integral values but may take any rational values, such as 1/2, 5/4, 3/17, -2, and so on.

In Section 4-2, we shall assume that (4) does give a satisfactory account of the growth of a colony of bacteria. We shall deduce an important consequence from this assumption and see how this consequence may be used to test the validity of equation (4) as a description of bacterial growth.

Of course if an experiment of this sort is repeated and if counts are taken at various times, it is too much to expect that the results will be fitted with complete accuracy by any simple equation like (4). A scientist needs a brief method of describing the results of his measurements. This method must fit the data with sufficient accuracy to serve as a convenient way of summarizing the facts and predicting the results of future experiments. A description of this sort furnishes a mathematical model of natural events. It is a fact that equations like (4) form satisfactory mathematical models for growth phenomena at least over limited periods of time.

We thus arrive by two quite different avenues at the idea of a function which associates with every rational number r a number $k2^r$ where k represents any real number, that is,

$$f:\ r \rightarrow k2^r$$

where r is a rational number.

For the sake of generality it is natural to ask for a way to extend the domain of f to include all real numbers x in such a way that the laws (1) and (2) will remain in force and so that the extended function $f: x \longrightarrow k2^x$ will have a smooth graph. The problem that this presents will become evident if you ask yourself what $2^{\sqrt{2}}$ or 2^{π} should be. We shall take up this extension in Section 4-3 and when it has been accomplished we shall have a new type of function which is defined for all real numbers x,

$$f: x \longrightarrow k2^x.$$

The number 2 occurs here as a result of the special example that we considered. We shall see that any positive number a may replace 2 so that we consider more general functions

$$f: x \longrightarrow ka^x$$

for all real numbers x, and a any positive real number. Such functions will be called exponential functions. Before going on we wish to show that the exponential functions are really functions of a new type and not just polynomial functions in a new dress. The proof which follows for $a = 2$ can easily be extended to a general value of a.

It is easy to show that 2^x is not a polynomial. In the first place, it is obvious that

$$f: x \longrightarrow 2^x$$

is not a constant function since

$$2^0 = 1 \quad \text{and} \quad 2^1 = 2.$$

Suppose then that 2^x has been defined for all real x and that

$$f(x) = 2^x = g_n(x) \qquad (5)$$

where $g_n(x)$ is a polynomial of degree $n > 0$.

Squaring both sides of (5) and using the fact that $(2^x)^2 = 2^{2x}$ we have

$$2^{2x} = \left[g_n(x)\right]^2.$$

But $2^{2x} = f(2x) = g_n(2x)$ if equation (5) is true. Hence

$$\left[g_n(x)\right]^2 = g_n(2x). \qquad (6)$$

However this is impossible since the degree of the term on the left of (6) is $2n$ while the degree of the term on the right is n, and $n \neq 2n$ if $n > 0$. Therefore the assumption that 2^x is a polynomial is false. We are consequently concerned with a new type of function which is the subject of the present chapter.

Exercises 4-1

1. If the identity (1) is to hold we must define 2^0 so that

$$2^0 \cdot 2^r = 2^{0+r} = 2^r.$$

Use this fact to show that 2^0 must equal 1.

2. Similarly, show that if we require that

$$2^{-r} \cdot 2^r = 2^0 = 1,$$

2^{-r} must be defined to be $1/2^r$.

3. Plot the graph of the equation

$$N = 10^6(2^n) \qquad \text{for } n = 1, 2, 3, 4$$

where N represents the number of bacteria present at the end of n days. (Note: The unit chosen for the N axis may be one million.)

Note: In Exercises 4 to 6 assume that the number of bacteria doubles in 24 hours.

4. The bacteria count at the end of $n + 5$ days is how many times as great as the count $n + 2$ days after the beginning of the experiment?

5. One week after the initial count was made the number of bacteria present was how many times as great as the number present three days before the experiment began?

6. If there are N bacteria present after 100 days, after how many days were there $N/4$ present?

7. Suppose that in a new experiment there are 200,000 bacteria present at the end of three days and 1,600,000 present at the end of 4 1/2 days. Compute:
   a) The number present at the end of 5 days;
   b) The number present at the end of 1 1/2 days;
   c) The number of days at the end of which there are 800,000 bacteria present.

   Hint: Assume that the number of bacteria present at the beginning of the experiment is N_0 and that at the end of 24 hours the count is $a \cdot N_0$.

4-2. Rational Powers of Positive Real Numbers

Let us assume (as in Section 4-1) that under favorable circumstances we can predict the number of bacteria in a certain colony by using the equation

$$N(x) = N_0 2^x \qquad (x \text{ rational}) \qquad (1)$$

for the number of bacteria x days after the start of the experiment. We then consider a bacteria count taken t days later where t is not necessarily a positive integer, but may by any rational number, positive, negative or zero. Then if (1) is indeed valid,

$$N(x + t) = N_0 2^{x+t}$$

$$= N_0 2^x \cdot 2^t$$

and

$$N(x + t) = 2^t N(x).$$

In other words, if $N(x)$ is the bacterial count at time x, the number of bacteria t days later, $N(x + t)$, is 2^t times as great. The multiplying factor 2^t does not depend upon x, the initial time, but only upon t, the time interval between counts.

Hence, no matter when we count bacteria, it is a consequence of equation (1) that t days later the number of bacteria will have increased in the ratio $r(t) = 2^t$ (where $r(t)$ depends only upon t).

Now, clearly, this consequence is easy to test experimentally. For example, we might count at 1/2 day intervals ($t = 1/2$). If (1) is correct, the ratio $\frac{N(1/2)}{N(0)}$ should be equal to the ratio $\frac{N(1)}{N(1/2)}$ or $\frac{N(3/2)}{N(1)}$. Within the limits of experimental error this is found to be true. We therefore feel justified in working with equations like (1) in studying bacterial growth.

To determine how many bacteria should be present at any particular time, we merely substitute the appropriate value of x in equation (1).

For example, let us suppose that $N_0 = 10^6$ (one million). The number of bacteria 1/2 day later should be

$$\begin{aligned} N(\tfrac{1}{2}) &= 10^6 \cdot (2^{1/2}) \\ &= 10^6 \sqrt{2} = 1.414(10^6), \text{ approximately.} \end{aligned}$$

After 3/2 days the number should be

$$\begin{aligned} N(\tfrac{3}{2}) &= 10^6 \cdot 2^{3/2} \\ &= 10^6(2 \sqrt{2}) \\ &= 2.828(10^6), \text{ approximately.} \end{aligned}$$

The number of bacteria 1 day before the initial count was taken should be

$$10^6 \cdot 2^{-1} = 500{,}000$$

assuming, of course, that the conditions of growth were the same prior to this count.

Let us recall and generalize our previous line of thought. We arrived at the equation

$$N(x) = N_0 2^x$$

from the basic assumption that one day after the start of the experiment (when $x = 1$) the initial number N_0 was multiplied by 2. The multiplication factor might obviously turn out to be different from 2, as we have seen in Exercises 4-1, Problem 7. It is quite natural therefore to consider more generally

$$N(x) = N_0 a^x \qquad (2)$$

where a is any real number > 0. This equation is obtained from (1) by replacing 2 by a. The number a is called the base of the exponential function. If $N(x)$ increases, $a > 1$.

Let us review briefly how we deal with numbers of the form $a^{m/n}$ where m and n are integers $(n \neq 0)$.

Positive integral exponents are elementary. Thus

$$a^2 = aa, \quad a^3 = aaa, \quad a^4 = aaaa,$$

and so on.

We next define $a^0 = 1$, $a^{-1} = 1/a$, $a^{-2} = 1/a^2$, and so on. If n is a positive integer, we define $a^{1/n} = \sqrt[n]{a}$, the positive n-th root of a.

Finally, if as before n is a positive integer and m is any integer, we define

$$a^{m/n} = (a^{1/n})^m = [\sqrt[n]{a}]^m.$$

It may be shown that

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m} .$$

It follows from these definitions that

$$a^r a^s = a^{r+s} \qquad (3)$$

and

$$(a^r)^s = a^{rs} \qquad (4)$$

where r and s are any rational numbers. We include a few exercises to familiarize you with these well-known definitions and laws.

Exercises 4-2a

1. Prove that

$$a^{m+n} = a^m \cdot a^n$$

for all integers m and n, positive, negative or zero, where a is any real number > 0.

2. Prove that

$$a^{mn} = (a^m)^n$$

for all integers m and n, where a is any real number > 0.

3. Evaluate: $1000(8^{-2/3})$, $3(\frac{9}{4})^{-3/2}$.

4. Arrange the following in order of magnitude:

$2^{2/3}$, $(4^{5/2})(8^{-1})$, $(\frac{1}{2})^{-4/3}$, 2^{-3}, $(2^{-2/9})^9$.

5. Show that if $x = 2^{2.7}$, then $x = 4\sqrt[10]{128}$.

6. Find the value of m if:

a) $8^m = (2^3)^2$;

b) $8^m = 2^{(3^2)}$.

7. Find the value of m if:

a) $2^{(4^5)} = 16^m$;

b) $(2^4)^5 = 16^m$.

8. Show that

$$\frac{2^h + 2^{h+2b}}{2} = 2^h \cdot \frac{1 + 2^{2b}}{2}$$

holds for all rational numbers b and h.

Rational Values of 2^r. Table 4-2 gives rational powers of 2. Ordinarily it is sufficient to use the entries rounded to 3 decimal place accuracy. The table can be extended by applying identity (3). You are asked to do this in Exercise 4.

The following Examples show how to find 2^r for values of r not listed in the table.

Example 1. Use Table 4-2 to find the value of $2^{1.68}$.

Solution. We note that

$$2^{1.68} = 2^{(1+0.65+0.03)} = 2^1 \cdot 2^{0.65} \cdot 2^{0.03}$$

$$\approx 2(1.569)(1.021) = 3.204 \quad \text{(approximately)}.$$

Example 2. Use Table 4-2 to find the value of $2^{-0.37}$.

Solution. We write

$$2^{-0.37} = 2^{-1+0.63}$$

$$= \frac{2^{0.60+0.03}}{2} \qquad = \frac{(2^{0.60})(2^{0.03})}{2}$$

$$\approx \frac{(1.516)(1.021)}{2} = 0.774 \quad \text{(approximately)}.$$

Table 4-2. Values of 2^r

r	2^r	2^{-r}
.001	1.000 693 4	0.999 307 1
.005	1.003 471 7	0.996 540 2
.01	1.006 955 6	0.993 092 5
.02	1.013 96	0.986 23
.03	1.021 01	0.979 42
.04	1.028 11	0.972 66
.05	1.035 26	0.965 94
.10	1.071 77	0.933 03
.15	1.109 57	0.901 25
.20	1.148 70	0.870 55
.25	1.189 21	0.840 90
.30	1.231 14	0.812 25
.35	1.274 56	0.784 58
.40	1.319 51	0.757 86
.45	1.366 04	0.732 04
.50	1.414 21	0.707 11
.55	1.464 08	0.683 02
.60	1.515 72	0.659 75
.65	1.569 17	0.637 28
.70	1.624 50	0.615 57
.75	1.681 79	0.594 60
.80	1.741 10	0.574 35
.85	1.802 50	0.554 78
.90	1.866 07	0.535 89
.95	1.931 87	0.517 63
1.00	2.000 00	0.500 00

Exercises 4-2b

1. Calculate $2^{5/4}$:
    a) By using the data in Table 4-2;
    b) By noting that $2^{5/4} = 2 \cdot 2^{1/4} = 2\sqrt{\sqrt{2}}$.
2. By using the data in Table 4-2 calculate:
    a) $2^{1.15}$
    b) $2^{2.65}$
    c) $2^{0.58}$
    d) $2^{-0.72}$
3. With the aid of Table 4-2 compute:
    a) $8^{0.84}$
    b) $0.25^{-0.63}$
4. Extend Table 4-2 by completing the following table.

Table 4-2 (extended) Values of 2^r

r	2^r
-4.0	
-3.6	
-3.2	
-2.8	
-2.4	
-2.0	
-1.6	
-1.2	
—	
1.4	
1.8	
2.2	
2.6	
3.0	

5. Plot the points $(x, 2^x)$ for the rational values of x shown in Table 4-2 and Table 4-2 extended (Exercise 4 above).

4-3. Arbitrary Real Exponents

In the preceding section, we dealt with the meaning to be assigned to 2^x and a^x for x a rational number, say r. Can we give meaning to these expressions if x is irrational? To be specific, can we define the expression 2^x in a natural way for irrational values of x? We know that

$$f: r \rightarrow 2^r$$

is defined for all rational numbers r. We wish to extend this function to a function with domain the set of all real numbers x.

Let us consider a very concrete example. How should we define the number $2^{\sqrt{2}}$? Since $\sqrt{2}$ is irrational, the number $2^{\sqrt{2}}$ has no meaning if we limit ourselves only to the definitions given in Section 4-2. Our problem is to assign a value to the expression $2^{\sqrt{2}}$, and indeed to all expressions 2^x for real numbers x, that will be reasonable and will preserve the usual rules of exponentiation.

We begin with the observation that the function $f: r \rightarrow 2^r$ defined for all rational numbers r increases strictly as r increases. That is, if r and s are rational numbers and $r < s$, then $2^r < 2^s$. The proof is given at the end of the present section. It thus seems reasonable to require that this property be preserved when we define 2^x for irrational numbers x. Thus, for $x = \sqrt{2}$, and for all rational numbers r and s such that

$$r < \sqrt{2} < s \qquad (1)$$

we should like to have

$$2^r < 2^{\sqrt{2}} < 2^s. \qquad (2)$$

Obviously this places a severe restriction on the value we assign to $2^{\sqrt{2}}$ and, as we shall see, determines it completely. The ordinary decimal approximants to $\sqrt{2}$ give us a handy collection of r's and s's. Thus we know that

$$\begin{array}{lcccl} 1.4 & < & \sqrt{2} & < & 1.5 \\ 1.41 & < & \sqrt{2} & < & 1.42 \\ 1.414 & < & \sqrt{2} & < & 1.415 \\ 1.4142 & < & \sqrt{2} & < & 1.4143 \\ 1.41421 & < & \sqrt{2} & < & 1.41422 \end{array}$$

and so on. The inequalities (1) and (2) then show that $2^{\sqrt{2}}$ must satisfy the following set of inequalities:

$$\begin{array}{lcccl} 2^{1.4} & < & 2^{\sqrt{2}} & < & 2^{1.5} \\ 2^{1.41} & < & 2^{\sqrt{2}} & < & 2^{1.42} \\ 2^{1.414} & < & 2^{\sqrt{2}} & < & 2^{1.415} \\ 2^{1.4142} & < & 2^{\sqrt{2}} & < & 2^{1.4143} \\ 2^{1.41421} & < & 2^{\sqrt{2}} & < & 2^{1.41422} \end{array}$$

and so on.

We replace the rational powers of 2 appearing in the last set of inequalities by certain decimal approximations and arrive at the following estimates for $2^{\sqrt{2}}$:

$$\begin{array}{lcccccccl} 2.639 & < & 2^{1.4} & < & 2^{\sqrt{2}} & < & 2^{1.5} & < & 2.829 \\ 2.657 & < & 2^{1.41} & < & 2^{\sqrt{2}} & < & 2^{1.42} & < & 2.676 \\ 2.664 & < & 2^{1.414} & < & 2^{\sqrt{2}} & < & 2^{1.415} & < & 2.667 \\ 2.665 & < & 2^{1.4142} & < & 2^{\sqrt{2}} & < & 2^{1.4143} & < & 2.666 \end{array}$$

and so on. Thus, if (2) is to hold, we know $2^{\sqrt{2}}$ to 3 decimal places: $2^{\sqrt{2}} = 2.665 \ldots$.

The pinching down process that we use to estimate $2^{\sqrt{2}}$ is sketched in Figure 4-3a.

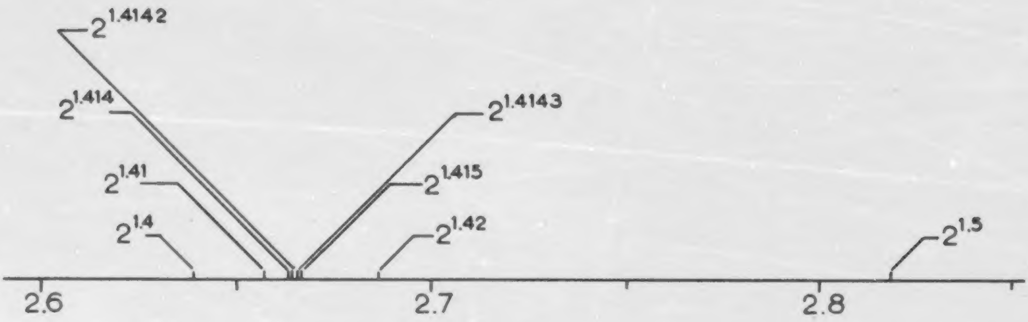


Figure 4-3a. Pinching down on $2^{\sqrt{2}}$.

To generalize to any real number x, we choose any increasing sequence $r_1, r_2, r_3, \ldots, r_n, \ldots$ of rational numbers all less than x, and any decreasing sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ of rational numbers all greater than x such that the difference $s_n - r_n$ can be made arbitrarily small. We compute the sequence of numbers $2^{r_1}, 2^{r_2}, 2^{r_3}, \ldots, 2^{r_n},$ and the sequence of numbers $2^{s_1}, 2^{s_2}, 2^{s_3}, \ldots, 2^{s_n}, \ldots$ and then look at the intervals $2^{r_n} \leq y \leq 2^{s_n}$.

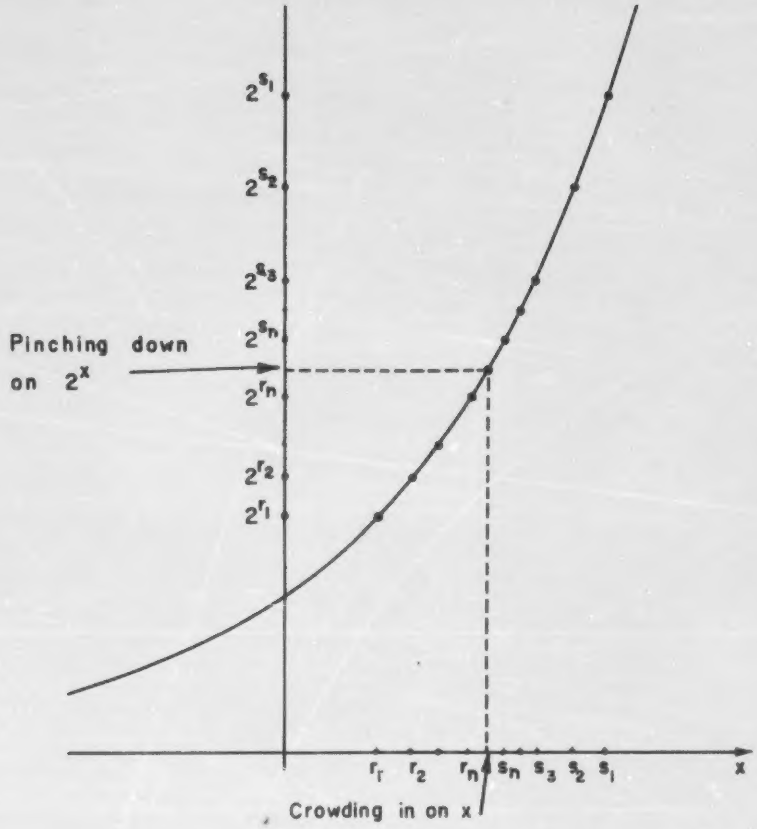

2^{s_1}
2^{s_2}
2^{s_3}
2^{s_n}
Pinching down on 2^x
2^{r_n}
2^{r_2}
2^{r_1}
r_1
r_2
r_n
s_n
s_3
s_2
s_1
x
Crowding in on x

Figure 4-3b. Pinching down on 2^x.

It is a property of the real number system that as x is confined by s_n and r_n to smaller and smaller intervals, the corresponding intervals on the y-axis pinch down to a uniquely determined number, which we shall define as the number 2^x. The number obtained is independent of the particular choice of the sequences $r_1, r_2, r_3, \ldots, r_n, \ldots$ and $s_1, s_2, s_3, \ldots s_n$.

The method used for defining 2^x for irrational x makes it possible to fill in all gaps in the graph of the function

$$r \longrightarrow 2^r \qquad (r \text{ rational})$$

to obtain a graph for

$$x \longrightarrow 2^x \qquad (x \text{ real}).$$

Figure 4-3c is a careful graph of this function for a limited part of its domain.

- -

Theorem. $2^r < 2^s$ if r and s are rational and $r < s$.

Proof. I. We first note that

(1) If $a > 1$, then $a^2 > 1$, $a^3 > 1$, ..., $a^n > 1$ (n a positive integer).

Similarly,

(2) If $a = 1$, then $a^n = 1$.

(3) If $0 < a < 1$, then $a^n < 1$.

II. We now assert that $2^{m/n} > 1$ for any positive integers m and n. For if $a = 2^{m/n}$ were equal to 1, (2) would lead to the result $a^n = 2^m = 1$, which is false. If, on the other hand, $2^{m/n}$ were less than 1, (3) would lead to the result $a^n = 2^m < 1$ which is also false. Since $2^{m/n}$ is neither less than nor equal to 1, it must be greater than 1.

III. Now let r and s be any two rational numbers such that $r < s$. Then $s - r$ is a positive rational number m/n. Since

$$2^{s-r} = 2^{m/n} > 1$$

$$2^r(2^{s-r}) > 2^r$$

or

$$2^s > 2^r .$$

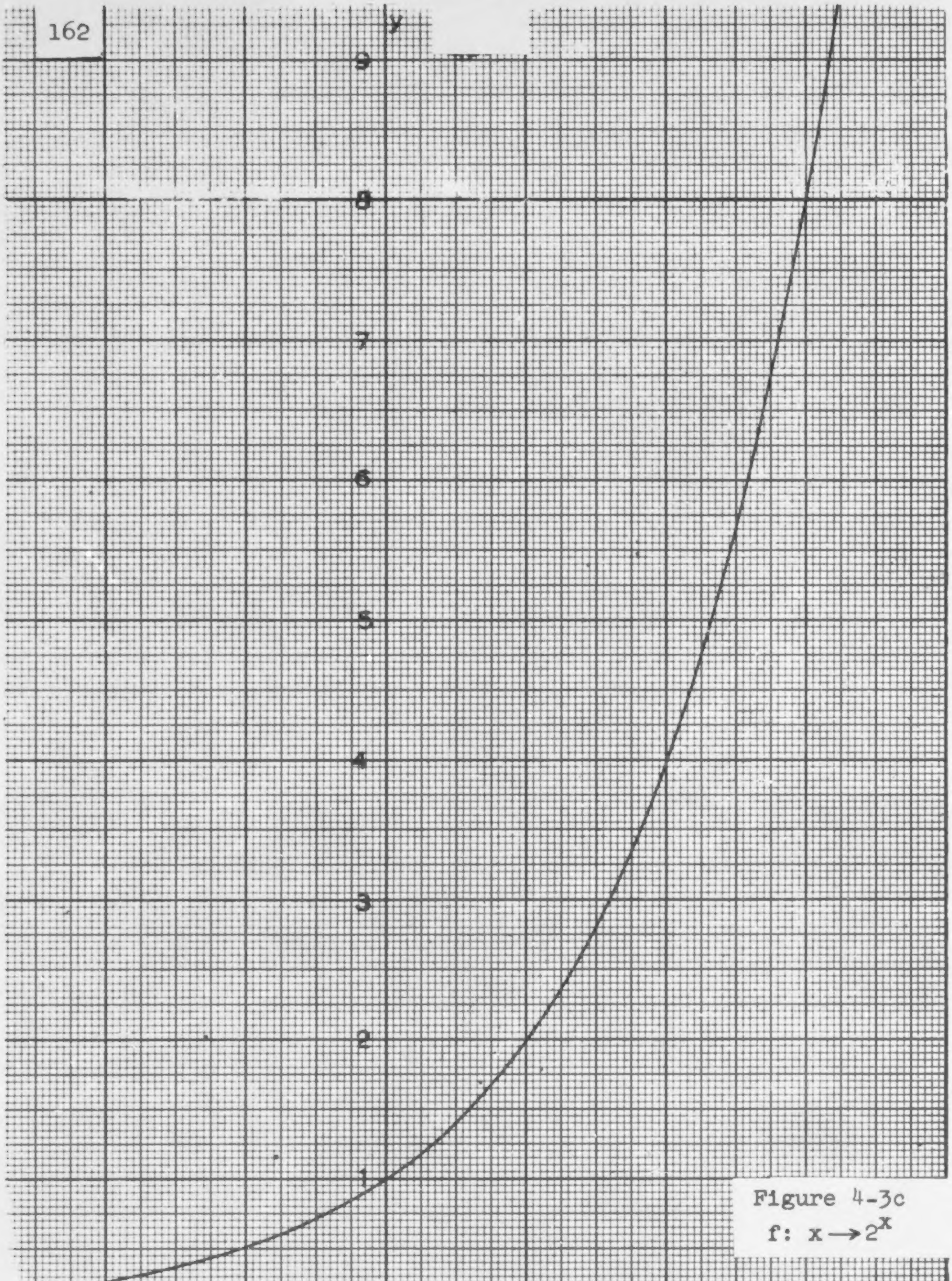


Figure 4-3c
f: $x \rightarrow 2^x$

Exercises 4-3

1. Use the graph of $x \longrightarrow 2^x$ to estimate the value of:
   a) $2^{1.15}$
   b) $2^{2.65}$
   c) $2^{0.58}$
   d) $2^{-0.72}$

2. Compare your results in Exercise 1 with your answers to Exercise 2 in the set 4-2b.

3. Use the graph of $x \longrightarrow 2^x$ to estimate the value of:
   a) $2^{\sqrt{3}}$
   b) 2^{π}
   c) $2^{-\pi/4}$

4. Is there any value of x for which $2^x = 0$? Give reasons for your answer.

5. Use the graph of $x \longrightarrow 2^x$ to estimate the value of x if:
   a) $2^x = 6$
   b) $2^x = 0.4$
   c) $2^x = 3.8$
   d) $2^x = 3$
   e) $2^x = 2.7$

4-4. Powers of the Base a as Powers of 2.

We have concentrated on the function

$$f:\ x \longrightarrow 2^x.$$

We are familiar with its graph and we have worked with a table of its values.

We shall need to study the function

$$f:\ x \longrightarrow a^x$$

where a is any positive real number. Fortunately we do not have to start from scratch because we can express a as a power of 2, as we proceed to show.

The graph of f: $x \rightarrow 2^x$ lies above the x-axis and rises from left to right. Also, $f(x) = 2^x$ becomes arbitrarily large for x sufficiently far to the right on the real number line, and arbitrarily close to zero for all x sufficiently far to the left on the real line. The graph has no gaps. Consequently, if we proceed from left to right along the graph, 2^x increases steadily in such a way that any given positive number a will be encountered once and only once. That is, there must be one and only one value of x, say α, for which

$$2^{\alpha} = a \qquad (1)$$

(See Figure 4-4) and therefore a may be expressed as a power of 2.

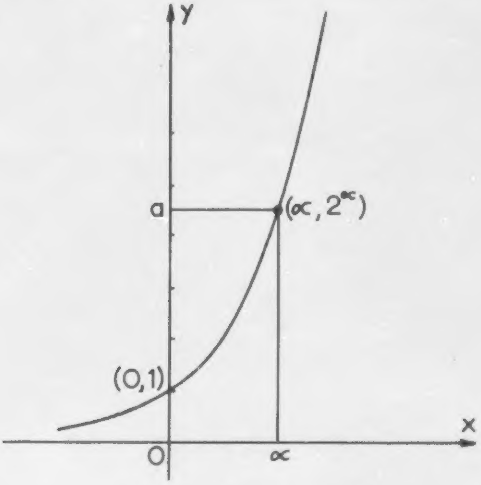


Figure 4-4. Graph of $x \rightarrow 2^x$ showing that $2^{\alpha} = a$.

We can find the value of α by means of the graph (Figure 4-3c) or Table 4-2.

Example 1. Find the value of α for which $2^{\alpha} = 6$.

Solution. On the graph of $x \rightarrow 2^x$ we look for the abscissa corresponding to the ordinate 6. The result is 2.6 (approximately).

If we use Table 4-2 to express 6 as a power of 2, we first write $6 = 2^2(1.5)$. Interpolating in Table 4-2 between the entries for $x = 0.55$ and 0.60 we obtain $2^{0.59} = 1.50$ (approximately). Hence, $6 = 2^2(1.50) \approx 2^2(2^{0.59}) = 2^{2.59}$ (approximately).

Example 2. Find the value of α for which $1.11 = 2^{\alpha}$.

Solution. We look for 1.11 in the second column and read backward to find the corresponding value of α in the first column. Thus, $1.11 = 2^{0.15}$ (approximately).

Example 3. Express 3.25 in the form 2^{α}.

Solution.

$$3.25 = 2(1.625) \approx 2^1(2^{0.70}) = 2^{1.70}.$$

To compute a^x for a given base a and given x we use equation (1) to write

$$a^x = (2^{\alpha})^x = 2^{\alpha x}$$

and then use Table 4-2 as illustrated in the following examples.

Example 4. Express $3^{0.7}$ as a power of 2, and find the approximate value of $3^{0.7}$.

Solution. To find the value of $3^{0.7}$ we first express 3 as a power of 2. Thus, $3 = 2^1(1.5) = 2^1(2^{0.59}) = 2^{1.59}$ (approximately). (Verify this from Figure 4-3c.)

Now
$$\begin{aligned} 3^{0.7} &\approx (2^{1.59})^{0.7} = 2^{1.113} \approx 2^{1.11} \\ &\approx 2^{(1+0.10+0.01)} = 2^1(2^{0.10})(2^{0.01}) \\ &\approx 2(1.072)(1.007) \approx 2.159. \end{aligned}$$

Example 5. Calculate the value of $(6.276)^{0.4}$

Solution. We note that

$$6.276 = 4(1.569) = 2^2(1.569) \approx 2^2(2^{0.65}) = 2^{2.65}$$

(Verify this from Figure 4-3c.)

Hence,

$$\begin{aligned}(6.276)^{0.4} &\approx (2^{2.65})^{0.4} = 2^{(2.65)(0.4)} = 2^{1.06}\\ &\approx 2^{(1+0.05+0.01)} = 2^1 \cdot 2^{0.05} \cdot 2^{0.01}\\ &\approx 2(1.035)(1.007) = 2.084 \quad \text{(approximately).}\end{aligned}$$

Exercises 4-4

1. Express 3.4 in the form 2^{α}.
2. Write 2.64 in the form $2.64 = 2^{\alpha}$ and then find the approximate value of $(2.64)^{0.3}$.
3. Find the approximate value of $(6.276)^{-0.6}$.
4. Find the approximate value of $(5.2)^{2.6}$.
5. Show that if $0 < a < 1$ and $v > u$, then $a^v < a^u$.

   Hint: See the proof of the theorem on page 161.

4-5. A Property of the Graph of $x \rightarrow 2^x$

The graph of f: $x \rightarrow 2^x$ has a simple but important property which is usually described by the phrase "concave upward". (See Figure 4-5a.) Precisely, this phrase means that if two points P and Q on the graph are joined by a straight line segment, then this segment lies above the arc PQ of the graph.

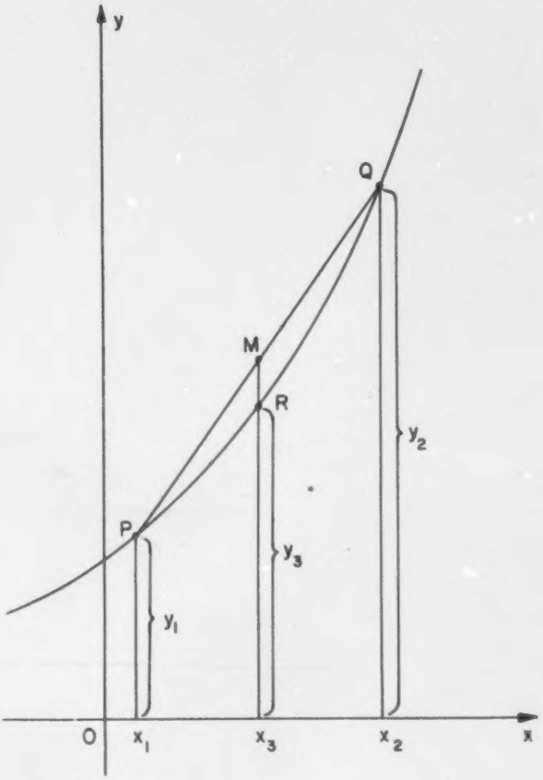


Figure 4-5a. The graph of $x \rightarrow 2^x$ is concave upward.

We shall not give a rigorous proof of this fact, but shall be content to show that the midpoint M of the segment PQ lies above the point R on the graph with the same abscissa.

Proof: Let P and Q have the coordinates (x_1, y_1) and (x_2, y_2) respectively. Then the ordinate of M is $\frac{y_1 + y_2}{2}$, the arithmetic mean of y_1 and y_2. If the ordinate of R is y_3,

$$\frac{y_3}{y_1} = \frac{y_2}{y_3} .$$

Hence $y_3 = \sqrt{y_1 y_2}$, the geometric mean of y_1 and y_2.

An important theorem states that the arithmetic mean of any two positive numbers is greater than or equal to their geometric mean.

In our case, this theorem would guarantee that

$$\frac{y_1 + y_2}{2} \geq \sqrt{y_1 y_2} \qquad (1)$$

The result that we require

$$\frac{y_1 + y_2}{2} > \sqrt{y_1 y_2} \qquad (2)$$

strengthens (1) by removing the equal sign. We shall prove (2) by replacing it by a list of equivalent inequalities, the last of which is obviously correct.

$$y_1 + y_2 > 2\sqrt{y_1 y_2}$$

$$y_1^2 + 2y_1y_2 + y_2^2 > 4y_1y_2$$

$$y_1^2 - 2y_1y_2 + y_2^2 > 0$$

$$(y_2 - y_1)^2 > 0 . \qquad (3)$$

Since $y_2 > y_1$, (3) holds so that (2) has been established.

Although our proof shows only that the midpoint M of the segment PQ lies above the arc, it is plausible that all points of the segment PQ lie above the corresponding points on the arc PQ. For we can apply our result to each of the arcs PR and RQ (see Figure 4-5b) and conclude that R_1 lies below M_1 and

R_2 below M_2. The process of bisection can be repeated as many times as we wish. We can therefore obtain an arbitrarily large number of points on the arc PQ which are certainly below the chord.

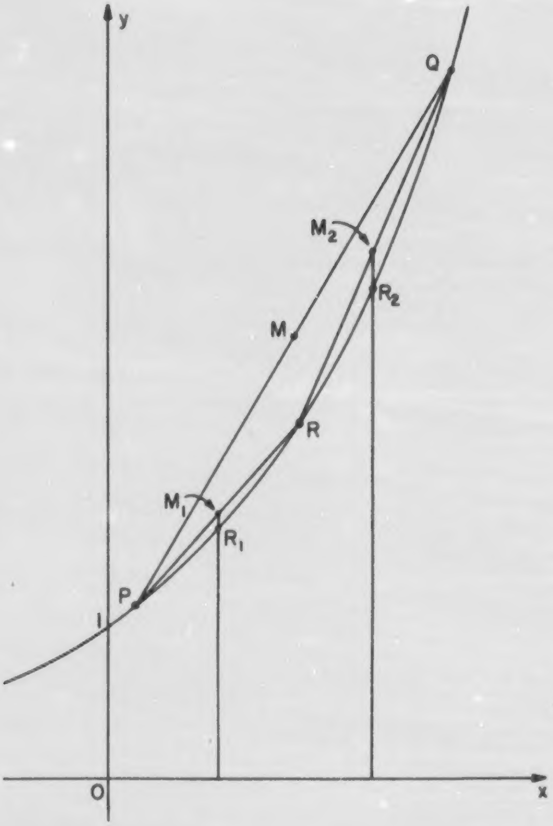


Figure 4-5b. Repeated bisection.

Exercises 4-5

1. Let P and Q have coordinates $(.05, 2^{.05})$ and $(.25, 2^{.25})$, respectively. Show that the midpoint M of the segment PQ lies above the point R on the graph G of $x \longrightarrow 2^x$ with

the same abscissa. Use Table 4-2.

2. Using the information given in Exercise 1, show that the midpoints M_1 and M_2 of segments PR and RQ respectively, lie above the corresponding points on G having the same abscissas. In this way exhibit 3 points on arc PQ which are below chord PQ.

3. Represent the results obtained in Exercises 1 and 2 graphically distorting the scale if necessary.

4. Repeat Exercises 1-3 using points $P(.05, 4^{.05})$ and $Q(.25, 4^{.25})$ on the graph G of $x \rightarrow 4^x$.

5. Generalize (1) to the case of four positive numbers, that is, prove that $\frac{y_1 + y_2 + y_3 + y_4}{4} \geq \sqrt[4]{y_1y_2y_3y_4}$.

   Hint: Use (1) and an analogous inequality for y_3 and y_4. Apply the principle to these two results.

*6. Generalize (1) to the case of three positive numbers, that is, prove that $\frac{y_1 + y_2 + y_3}{3} > \sqrt[3]{y_1y_2y_3}$.

   Hint: Use Exercise 5, replacing y_4 by $\frac{y_1 + y_2 + y_3}{3}$, and simplify.

4-6. Tangent Lines to Exponential Graphs

In Chapter 3 we learned that through each point P of a polynomial graph there exists a certain straight line, the tangent, that represents the best linear approximation to the graph near P. One may therefore be led to wonder whether a similar statement applies to the graph of an exponential function. It is in fact true that there does exist a tangent at each point of the graph.

In the present section we shall show how to find the slopes of such tangent lines. The results obtained have important applications, specifically to problems of growth and of radioactive decay.

We shall begin with the graph G of $g: x \to 2^x$ and its tangent at (0, 1).

Let $Q(b, 2^b)$ be a point on the graph to the right of P and let R be the point $(-b, 2^{-b})$ (See Figure 4-6a). We shall assume that $b < 1$.

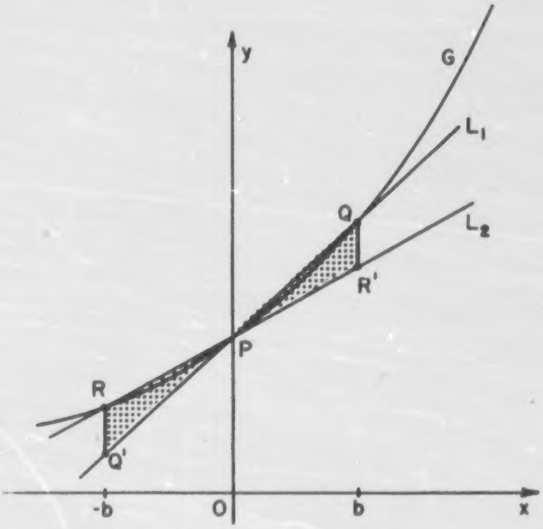


Figure 4-6a. Approximating the graph of $x \to 2^x$ near P(0,1)

As we know, G is concave upward everywhere; hence G lies below PQ and below RP.

Let us extend QP to Q' so that Q'P = PQ, and RP to R' so that PR' = RP. We shall show that for $|x| < b$, G lies in the hatched region between the lines $L_1 = QQ'$ and $L_2 = RR'$.

It is necessary to prove that G lies above L_1 (slope m) for $-b < x < 0$, that is, that $2^x > 1 + mx$, $-b < x < 0$. This is equivalent to the statement that $2^{-x} > 1 - mx$ for $0 < x < b$. Now $2^{-x} = \frac{1}{2^x}$ and $2^x < 1 + mx$, $0 < x < b$. Hence

$$2^{-x} = \frac{1}{2^x} > \frac{1}{1 + mx} \quad . \qquad (1)$$

Then $2^{-x} > \frac{1 \cdot (1 - mx)}{(1 + mx)(1 - mx)} = \frac{1 - mx}{1 - m^2x^2}$.

Since $1 - m^2x^2 < 1$ (2)

$$2^{-x} > 1 - mx \tag{3}$$

which is the required conclusion.

It can similarly be proved that for $0 < x < b$, G lies above L_2.

We expect that if b is small enough, the lines L_1 and L_2 will have slopes which differ as little as we please. This is indeed the case.

Suppose for example that $b = 0.01$. Since

$$2^{0.01} \approx 1.00696 \quad \text{and} \quad 2^{-0.01} \approx 0.99310,$$

we obtain for the slope of L_1, $\frac{0.00696}{0.01} = 0.696$,

and for the slope of L_2, $\frac{0.00690}{0.01} = 0.690$.

We could take $b = 0.001$ or even smaller, and thus get lines whose slopes are even closer together. It turns out that all of these wedges include a line whose slope to 6 decimal places is 0.693147. We shall use the letter k to stand for this slope, and write the equation of the tangent at P as

$$y = kx + 1,$$

where $k \approx 0.693$ is a sufficiently good approximation for most purposes. Thus if

$$g: x \longrightarrow 2^x,$$

$$g(x) \approx kx + 1, \quad \text{for } |x| \text{ small.} \tag{4}$$

Exercises 4-6a

1. Consider the proof in the text that the graph G of $x \to 2^x$ lies above L_1, for $-b < x < 0$.
   At which step is it necessary to assume that $mx \neq -1$? $mx \neq 0$? $mx \neq +1$?
   Show that none of these possibilities can occur.

*2. Prove that G lies above L_2 for $0 < x < b$.
   Hint: Note that G lies below L_2 for $-b < x < 0$; see the given proof that G lies above L_1, $-b < x < 0$.

3. Find the slopes of the lines L_1 and L_2 if $b = 0.001$.
   Hint: Use Table 4-2.

4. Using the results obtained in Exercise 3, show that the slope k of the tangent to the graph of $x \to 2^x$ at P(0, 1) lies between 0.6929 and 0.6934.

5. Using the same procedure as in Exercise 3 and 4 calculate the slope of the tangent to the graph of $x \to 4^x$ at P(0, 1). Compare your answer with the result obtained in Exercise 4.

6. Find the slope of the tangent to the graph of $x \to 3^x$ at P(0, 1).
   Note: Correct to 4 decimal places, the result is 1.0986.

So far we have considered only the base $a = 2$. It is easy however to obtain a general result for the slope of the tangent to the graph of

$$g: x \to a^x$$

at (0, 1) for any $a > 0$.

It is sufficient to write $a = 2^{\alpha}$ so that

$$a^x = 2^{\alpha x} = g(\alpha x)$$

From (4) it follows that

$$g(\alpha x) \approx k\alpha x + 1, \text{ for } |\alpha x| \text{ small,}$$

and hence the required tangent has the equation

$$y = k\alpha x + 1 \tag{5}$$

where α is such a number that $2^{\alpha} = a$.

It would be highly desirable to have the slope of (5) equal to 1. This result is obtained by taking $\alpha k = 1$, that is,

$$\alpha = 1/k.$$

This happy choice of the base a is universally denoted by the letter e.

Definition 4-1. $e = 2^{1/k}$, where $k = 0.693147\ldots$.

The use of e in this sense may be traced to the Swiss mathematician Leonard Euler (1707-1783). The number e is one of the most important in mathematics; it ranks in importance with π. If we use 0.693 as an approximation for k, we obtain

$$\frac{1}{k} \approx \frac{1}{0.693} \approx 1.443,$$

and
$$\begin{aligned} e = 2^{1/k} \approx 2^{1.443} &= 2(2^{0.4})(2^{0.04})(2^{0.003}) \\ &\approx 2(1.320)(1.028)(1.002) \\ &= 2.72 \text{ approximately.} \end{aligned}$$

If we use a closer approximation to k, we may naturally expect to get a better approximation to e.

The number e has been computed to an enormous number of decimal places. In recent years, high speed electronic digital computers have been used to obtain the decimal expansion of e to 2500 places. For the record, we note that the first 15 places are given by

$$e = 2.71828\ 18284\ 59045\ldots \tag{6}$$

For most purposes $e = 2.718$ is a sufficiently good approximation.

There is an important method for approximating the value of e, which may be expressed as follows

$$e \approx \left(1 + \frac{1}{n}\right)^{n} \quad \text{for } n \text{ large.} \tag{7}$$

This means that if n is a large positive integer, say 100, e is given approximately by $(1 + \frac{1}{n})^n$. This result may be made plausible by the following argument. We may expect that the tangent line to a given curve will lie very close to the curve itself for all points close to the point of tangency. Consider the graph of $f: x \rightarrow e^x$ and the point $(0, 1)$ lying on this graph. Since the slope of the graph at $(0, 1)$ is 1, the tangent line at $(0, 1)$ has the equation $y = 1 + x$. Thus we write

$$e^x \approx 1 + x \text{ for } |x| \text{ near zero.}$$

This being so, we take $1/n$ very small (n a large positive integer) and write

$$e^{1/n} \approx 1 + \frac{1}{n} .$$

It is indeed correct that for large n, (7) does give a good value for e. In fact by choosing n large enough, an arbitrarily close approximation may be obtained. A further discussion of methods for computing e will be found in Appendices 4-16 and 4-17. A table of values of e^x and e^{-x} has been included together with a graph of $y = e^x$ for $-2\frac{1}{4} < x < 2\frac{1}{4}$. See Table 4-6 and Figure 4-6b.

Table 4-6 e^x and e^{-x}

x	e^x	e^{-x}
0.00	1.0000	1.0000
0.01	1.0101	0.9901
0.02	1.0202	0.9802
0.03	1.0305	0.9704
0.04	1.0408	0.9608
0.05	1.0513	0.9512
0.10	1.1052	0.9048
0.15	1.1618	0.8607
0.20	1.2214	0.8187
0.25	1.2840	0.7788
0.30	1.3499	0.7408
0.35	1.4191	0.7047
0.40	1.4918	0.6703
0.45	1.5683	0.6376
0.50	1.6487	0.6065
0.55	1.7333	0.5770
0.60	1.8221	0.5488
0.65	1.9155	0.5220
0.70	2.0138	0.4966
0.75	2.1170	0.4724
0.80	2.2255	0.4493
0.85	2.3396	0.4274
0.90	2.4596	0.4066
0.95	2.5857	0.3867
1.00	2.7183	0.3679
1.50	4.4817	0.2231
2.00	7.3891	0.1353
2.50	12.182	0.0821
3.00	20.086	0.0498
4.00	54.598	0.0183
5.00	148.41	0.0067

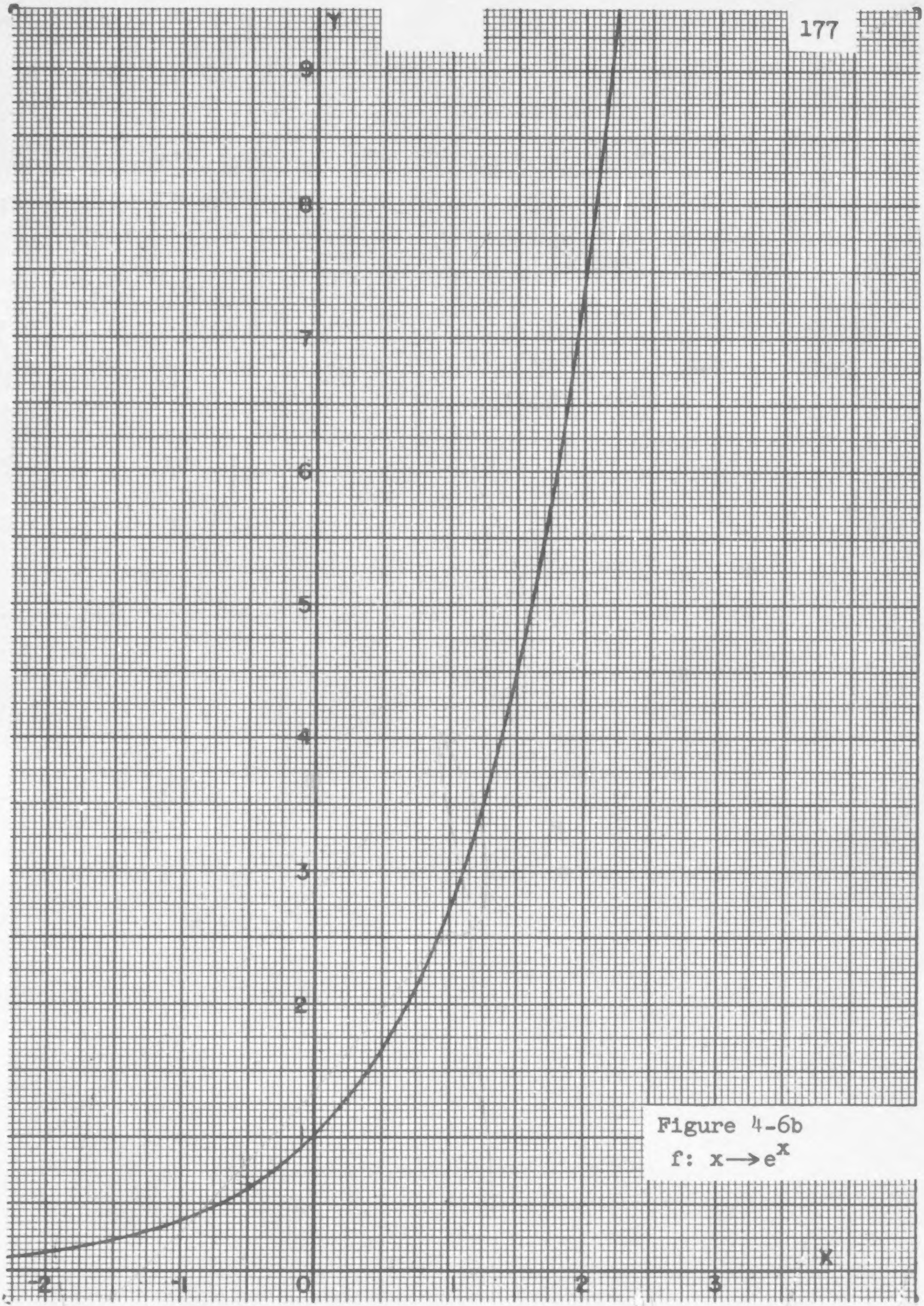


Figure 4-6b
f: $x \rightarrow e^x$

Exercises 4-6b

1. Show graphically that there is a unique real number x such that $e^x + x = 0$.
2. Using Newton's Method find the zero of $x \longrightarrow e^x + x$ correct to two decimal places.

4-7. Tangents to Exponential Graphs (Continued).

It remains to discuss the tangents to the graph of

$$x \longrightarrow a^x$$

at points that are not on the y-axis. We shall confine ourselves to the case $a = e$, where the base is e. As we know, the tangent at $(0, 1)$ has the equation $y = 1 + x$. Consequently, for $|x|$ small enough,

$$e^x \approx 1 + x. \qquad (1)$$

To study the graph near $P(h, e^h)$, we write $x = h + (x-h)$ and

$$e^x = e^{h+(x-h)} = e^h \cdot e^{x-h} \qquad (2)$$

For $|x - h|$ small enough, we use (1) and replace e^{x-h} by $1 + (x-h)$.

Then (2) becomes $e^x \approx e^h[1 + (x-h)] = e^h + e^h(x-h)$. This linear approximation gives the equation of the tangent at P, namely,

$$y = e^h + e^h(x - h).$$

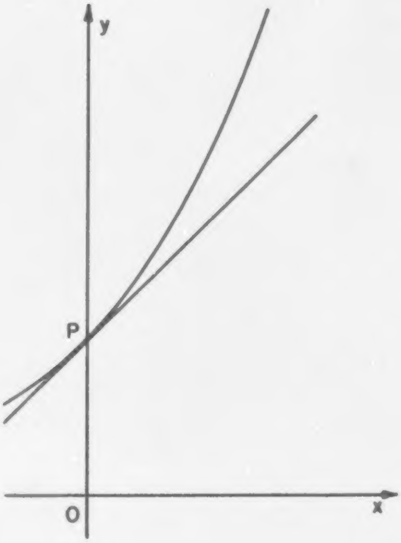


Figure 4-7

The graph of $x \longrightarrow e^x$ and its tangent at point (0, 1)

The significant part of the result is that the slope of the tangent is equal to the value, e^h, of the function. Otherwise expressed, if

$$f: x \rightarrow e^x,$$

then the slope function f' is

$$f': x \rightarrow e^x,$$

that is, $f' = f$.

Exercises 4-7

1. Use the data in Table 4-6 to find the slope of the tangent to the graph of $f: x \rightarrow e^x$ at the following points.
   a) $(-1, e^{-1})$
   b) $(0.5, e^{0.5})$
   c) $(0.7, e^{0.7})$
   d) $(0, 1)$
   e) $(1.5, e^{1.5})$
2. Use the graph of $f: x \rightarrow e^x$ in Figure 4-6b to estimate the slope of the tangent at the points given in Exercise 1. Compare your results with those obtained in Exercise 1.
3. Write an equation of the tangent to the graph of f at each point (x, e^x) given in Exercise 1.
4. a) Through the point $(3, 4)$ draw a line L_1 with slope $m = 2/5$.
   b) Draw a line L_2 which is symmetric to L_1 with respect to the y-axis.
   c) What point on L_2 corresponds to the point $(3, 4)$ on L_1?
   d) What is the slope of L_2?
   e) Consider the general case: line L_1 drawn through point (r, s) with slope $= m$, and line L_2 symmetric to L_1 with respect to the y-axis.
   What point on L_2 corresponds to point (r, s) on L_1?
   What is the slope of L_2?

5. a) Plot the points (x, e^x) for which $x = -2.0, -1.8, \ldots, 0.2, 0.4, \ldots, 1.6$.

   b) Through each of these points draw the graph of a line having slope $m = e^x$.

   c) Show that these lines suggest the shape of the graph of $f: x \rightarrow e^x$.

6. a) For each point plotted in Exercise 5(a) locate the corresponding point which is symmetric with respect to the y-axis; then through these points draw lines symmetric to those of Exercise 5(b) with respect to the y-axis.

   b) Show that each point located in 6(a) lies on the graph of $g: x \rightarrow e^{-x}$.

   c) Compare the slopes of the lines drawn in Exercise 6(a) with those of Exercise 5(b).

7. a) Using the same coordinate axes draw the graphs of $f: x \rightarrow e^x$ and $g: x \rightarrow e^{-x}$.

   b) Compare the slopes of the graphs drawn in (a) at $x = 0, +1, -1$.

   c) Compare the slope of the graph of g at $x = h$ with e^{-h}.

*8. Write an equation describing the slope function g' of $g: x \rightarrow e^{-x}$.

4-8. Applications.

In the early part of this chapter we examined the equation

$$N(x) = N_0 a^x, \quad a > 0, \tag{1}$$

If $a > 1$, (1) is a mathematical model for bacterial growth. Similar equations arise in many branches of science.

Radioactive decay. Radioactive substances have the property of disintegrating so that in a given period of time some of the atoms of a given radioactive substance break up by emitting particles, thus changing to atoms of another substance. As a result, the weight of the unchanged material decreases with time. The weight $W(x)$ of the radioactive material at time x is satisfactorily given by

$$W(x) = W_0 a^{-x} \qquad (2)$$

where W_0 is the weight $W(0)$ at $x = 0$ and a is a suitable constant greater than one.

The negative exponent corresponds to the fact that $W(x)$ decreases with time.

We may write equation (2) using the base 2.

Thus, if $a = 2^{\alpha}$

$$W(x) = W_0 2^{-\alpha x} . \qquad (3)$$

We may also use the base $e = 2^{1/k}$. Since $2 = e^k$, $2^{\alpha} = e^{k\alpha}$. Substituting in (3) we have

$$W(x) = W_0 e^{-k\alpha x}$$

or

$$W(x) = W_0 e^{-cx} , \qquad (4)$$

where $c = k\alpha$.

The fraction of radioactive material which remains after a given interval of length t is fixed, since

$$\frac{W(x+t)}{W(x)} = \frac{W_0 a^{-(x+t)}}{W_0 a^{-x}} = a^{-t}$$

is independent of x.

The ratio $\frac{W(x+t)}{W(x)}$ is less than 1 for all values of t, since $t > 0$ and $a > 1$.

Workers in the field of radioactivity usually measure the rate of radioactive decay of an element in terms of its half-life. The half-life is the time required for one-half of the active material present at any time to disintegrate. The half-life of radioactive bismuth (Radium E), for example, is 5.0 days. At the end of 10.0 days three-fourths of it will have disintegrated, leaving only one-fourth of the initial amount. At the end of 20.0 days, only one-sixteenth of the original radioactive bismuth is present.

To find the half-life T of a given radioactive substance it is convenient to use equation (3). Thus, the weight $W(x)$ at time T equals one-half the initial weight W_0. We find accordingly that

$$\frac{1}{2} W_0 = W_0 2^{-\alpha T}$$

and

$$2^{-1} = 2^{-\alpha T} .$$

Hence

$$\alpha T = 1 \quad \text{and} \quad \alpha = \frac{1}{T} .$$

Thus the process of radioactive decay of an element is described completely by the equation

$$W(x) = W_0 2^{-x/T} . \qquad (5)$$

Example 1. If radium decomposes in such a way that at the end of 1620 years one-half of the original amount remains, what fraction of a sample of radium remains after 405 years?

Solution. The fraction remaining after x years is

$$\frac{W(x)}{W(0)} .$$

From the given data we have

$$x = 405,$$
$$T = 1620.$$

Equation (5) gives us

$$W(405) = W_0 \cdot 2^{-405/1620} = W_0 2^{-1/4} .$$

Hence the fraction remaining after 405 years is

$$\frac{W(405)}{W(0)} = 2^{-1/4} = \frac{1}{\sqrt[4]{2}} \approx 0.841.$$

Example 2. Find the half-life of uranium if 1/3 of the substance decomposes in 0.26 billion years.

Solution. From the given data

$$W(0.26) = \frac{2}{3} W_0.$$

Substituting in equation (5) we have

$$\frac{2}{3} W_0 = W_0 2^{-0.26/T}$$

and hence, $2^{-0.26/T} = \frac{2}{3}$, or $2^{0.26/T} = 1.5$.

From Table 4-2

$$2^{0.58} = 1.5 \quad \text{(approximately)}.$$

Thus

$$\frac{0.26}{T} \approx 0.58 ,$$

so that

$$T \approx 0.45 \text{ (billion years)}$$

This means that no matter what amount of uranium is present at any given time, 4.5×10^8 years later one-half of it will have decayed.

Exercises 4-8a

1. The half-life of radon is 3.85 days. What fraction of a given sample of radon remains at the end of 7.7 days? After 30.8 days?
2. The half-life of radioactive lead is 26.8 minutes. What fraction of a sample of lead remains after a time of 13.4 minutes? After 80.4 minutes?
3. At the end of 12.2 minutes, 1/16 of a sample of polonium remains. What is the half-life?
4. A certain radioactive substance disintegrates at such a rate that at the end of a year there is only 49/50 times as much

as there was at the beginning of the year. If there are two milligrams of the substance at a certain time, how much will be left t years later?

5. A quantity of thorium has decreased to 3/4 of its initial amount after 3.36×10^4 years. What is the half-life of thorium measured in years?

6. Radium decomposes in such a way that, of m milligrams of radium, $0.277\,m$ milligrams will remain at the end of three thousand years. How much of 2 milligrams will remain after 81 decades?

Compound interest. Suppose that P dollars is invested at an annual rate of interest of r per cent or $r/100$, and at the end of each year interest is compounded, or added to the principal. After t years the total amount A_t on hand is given by

$$A_t = P(1 + r/100)^t.$$

However, the interest may be compounded semiannually, quarterly, or n times a year. If interest is added to the principal n times per year, the rate of interest is $\frac{r}{100n}$ per period, and the number of periods in t years is nt. Hence, the amount A_{nt} after nt periods (that is, after t years) is

$$A_{nt} = P\left(1 + \frac{r}{100n}\right)^{nt}. \qquad (6)$$

The more often you compound interest, the more complicated the calculation becomes. On the other hand, if we let n in (6) get larger and larger indefinitely, we approach the theoretical situation in which interest is compounded continuously; we shall see that the result obtained will enable us to find easily a very satisfactory approximation for the amount of money on hand at the end of a reasonable period of time.

To study this idea, let $\frac{r}{100n} = h$ so that $n = \frac{r}{100h}$. Then (6) becomes

$$A_{nt} = P(1 + h)^{rt/100h}$$

$$= P\left[(1 + h)^{1/h}\right]^{rt/100} . \qquad (7)$$

If now n is taken larger and larger, h gets smaller and smaller and the right side of (7) grows closer and closer to

$$Pe^{rt/100}$$

which we call A, the theoretical amount that would be obtained if interest were compounded continuously at r per cent. Thus

$$A = Pe^{rt/100}. \qquad (8)$$

Example. If $100 is invested at 4 per cent for 10 years, compare the amount after 10 years when interest is compounded continuously with the amount after 10 years if interest is compounded only annually.

Solution. We have $P = 100$, $r = 4$, and $t = 10$ (years). If interest is compounded continuously, (8) gives

$$A = 100e^{0.4} ,$$

which is approximately 149.

To compute interest compounded annually we substitute the above values of P, r, and t in (6). This gives

$$A_{10} = 100(1.04)^{10} .$$

We may use a table of common logarithms to estimate A_{10}; thus

$$A_{10} \approx 100(1.48) = 148.$$

The results, $149 and $148, differ surprisingly little.

Exercises 4-8b

1. Find the amount of \$1000 after 18 years if interest is compounded continuously at the rate of 3 per cent.
2. Using $2 \approx e^{0.693}$, find how many years it takes to double a sum of money at
   a) 3 per cent compounded continuously.
   b) 6 per cent compounded continuously.
   c) n per cent compounded continuously.

A Law of Cooling. The temperature of a body warmer than the surrounding air decreases at a rate which is proportional to the difference in temperature between the body and the surrounding air. Let $T(x)$ denote the temperature of the body and B the temperature of the air at time x. The law of cooling may be expressed by

$$T(x) - B = Ae^{-cx}$$

or

$$T(x) = B + Ae^{-cx} \qquad (9)$$

In this equation $A + B$ is the temperature of the body at time 0 and c is a positive constant whose value determines the rate at which cooling takes place.

If we let $T(x) - B = W(x)$ and $A = W(0)$, we have

$$W(x) = W(0)e^{-cx}$$

which is identical with our previous equation (4).

Example 1. A kettle of water has an initial temperature of 100° C. The room temperature is 20° C. After 10 minutes, the temperature of the kettle is 80° C.

a) What is the temperature after 20 minutes?

b) When will the temperature be 40° C?

Solution. Since $W(x) = T(x) - 20$

$$W(0) = 100 - 20 = 80$$

and $$W(10) = 80 - 20 = 60.$$

From equation (4)

$$W(x) = 80e^{-cx}$$

$$\text{and} \quad 60 = 80e^{-c10} .$$

Then $e^{-10c} = \frac{60}{80} = .75 \approx e^{-.30}$ (from Table 4-6),

so that

$$10c \approx .30 \quad \text{and} \quad c \approx .03.$$

Hence $\quad W(x) = 80e^{-.03x}$

$$\text{and} \quad T(x) = 20 + 80e^{-.03x}.$$

(a) $\quad T(20) = 20 + 80e^{-.6}$

$$\approx 20 + 80(.549)$$

$\approx 20 + 43.9 = 63.9$; hence the temperature after 20 minutes is about 64° C.

(b) $\quad 40 = 20 + 80e^{-.03x}$

$$e^{-.03x} = \frac{20}{80} = .25$$

$$.03x \approx 1.39$$

$x \approx 46.3$; hence the temperature will be 40° C after about 46 minutes.

Example 2. The law of cooling can be applied to solve the problem of whether to put your cream in your coffee at once or to add the cream just before drinking it. Suppose that you are served a cup holding, let us say, 6 ounces of coffee at temperature 180° F. You are also supplied with one ounce of cream which is at room temperature, 70° F. You wish to wait for a while before drinking the coffee and also wish to have it as hot as possible when you drink it. How can you get the hottest coffee? Should you pour the cream in right away, or wait until you are ready to drink it? We solve this problem easily if we assume that the exponential law of cooling holds.

First case. Pour the cream in right away. Then the temperature of the mixture becomes

$$180(\tfrac{6}{7}) + 70(\tfrac{1}{7}) = \frac{660}{7} + 70$$

since 6/7 of the mixture (coffee) is at temperature 180° and 1/7 (cream) is at 70°. Using equation (9) with $B = 70$ and initial temperature $A + B = 660/7 + 70$ we find that the temperature at time x is

$$T(x) = \frac{660}{7} e^{-cx} + 70. \tag{10}$$

Second case. Pour the cream in at any time you wish. Then at time x, just before pouring in the cream, the coffee has cooled to temperature

$$70 + 110e^{-cx}$$

according to (9) with $B = 70$, $A + B = 180$. Mixing the coffee and cream now, we find that its temperature is

$$T(x) = (70 + 110e^{-cx})(\tfrac{6}{7}) + 70(\tfrac{1}{7}).$$

Reducing this expression by elementary algebra gives

$$T(x) = \frac{660}{7} e^{-cx} + 70 \tag{11}$$

which is the same as the result obtained in (10). Therefore it makes no difference at all whether you pour your cream into your coffee as soon as you get it, or wait and pour it in just before drinking. It will be as cold one way as the other.

Exercises 4-8c

1. At h kilometers above sea level, the pressure in millimeters of mercury is given by the formula

$$P = P_0 e^{-0.11445h}$$

where P_0 is the pressure at sea level. If $P_0 = 760$, at what height is the pressure 180 millimeters of mercury?

2. A law frequently applied to the healing of wounds is expressed by the formula

$$Q = Q_0 e^{-nr},$$

where Q_0 is the original area of the wound, Q is the area that remains unhealed after n days, and r is the so-called rate of healing. If $r = 0.12$, find the time required for a wound to be half-healed.

3. If in a room of temperature 20° C a body cools from 100° C to 90° C in 5 minutes, when will the temperature be 30° C? (Assume the law of cooling expressed by (9)).

4. If light of intensity I_0 falls perpendicularly on a block of glass, its intensity I at a depth of x feet is

$$I = I_0 e^{-kx} .$$

If one third of the light is absorbed by 5 feet of glass, what is the intensity 10 feet below the surface? At what depth is the intensity $1/2\ I_0$?

4-9. Inversion

Before we can proceed with the main ideas of this chapter, we must return to the concept of an inverse function, which was introduced in Section 1-6. We shall explore this idea a little further than we did in Chapter 1, and also prove some of the most important theorems which relate to inverses. We start by repeating the definition of inverse.

Definition 1-8. If f and g are functions that are so related that $(fg)(x) = x$ for every element x in the domain of g and $(gf)(y) = y$ for every element y in the domain of f, then f and g are said to be inverses of each other. In this case both f and g are said to have an inverse, and each is said to be an inverse of the other.

This definition leaves unanswered one important question: Can a function have more than one inverse? That is, if f and

g are inverses of each other, does there exist a function $h \neq g$ such that f and h are also inverses of each other? As you might suspect, the answer is no, but we shall not prove it here. Consider, however, a picture of a function as a mapping, with arrows going (as in Figure 1-6a) from points representing elements of the domain to points representing elements of the range. To represent the inverse function, we merely reverse the direction of each arrow, as in Figure 1-6b. It seems intuitively clear that there is only one way to do this.

The fact that a function can have at most one inverse justifies our use of a distinctive notation for functions which are inverses of each other. If f and g are such functions, then we can say that g is the inverse of f and write $g = f^{-1}$. We read f^{-1} as "f inverse". Similarly we can write $f = g^{-1}$. Thus $(f^{-1})^{-1} = f$.

Warning. Although the notation f^{-1} is strongly suggestive of "1 divided by f," it has nothing whatever to do with division. All it means is that

$$(ff^{-1})(x) = x \qquad \text{and} \qquad (f^{-1}f)(y) = y.$$

We now prove the basic theorems which relate to the existence of inverses.

Theorem 4-1. If a function f has an inverse then $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 are two distinct elements of the domain of f.

Proof. We shall prove this theorem by assuming the contrary and then deriving a contradiction. Hence we assume that $f(x_1) = f(x_2)$. From this we see that $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$. Now $f^{-1}f(x_1) = x_1$ and $f^{-1}f(x_2) = x_2$, so it follows that $x_1 = x_2$. But the elements x_1 and x_2 are supposed to be distinct (i.e., $x_1 \neq x_2$). This contradiction proves the theorem.

A vivid expression is used to describe functions f for which $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. This is the expression "one-to-one". If a function has an inverse then by Theorem 4-1 it is one-to-one. Note that in this case the equation $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

We point out that the idea of a one-to-one function is fundamental to the process of counting a collection of objects. When we count a set of things we associate the number 1 with one of the things, the number 2 with another, and so on until all the objects have been paired off with whole numbers. We do not give the same number to two different objects in the collection. In short, this "counting" function is one-to-one. As another example, suppose that there are 300 seats in a theater, and suppose that each seat is occupied by one and only one patron. Then, without counting the people, we can conclude that there must be 300 people sitting in these seats. These two examples deal with finite sets. On the other hand, the idea of a one-to-one function is fruitful even when the sets involved are not finite. Indeed, most of the applications deal with sets of this kind.

Now that we know that every function which has an inverse is one-to-one, it is natural to ask if the converse is true. Does every one-to-one function have an inverse? You might guess that the answer is yes. This is the content of Theorem 4-2.

Theorem 4-2. If f is a function which is one-to-one then f has an inverse.

Proof. Using the hypothesis that f is one-to-one, we shall construct a function which will turn out to be f^{-1}. Given an element y of the range of f, then, since f is one-to-one, there exists one and only one element x in the domain of f such that $y = f(x)$. Now associate the element x with the element y. This association defines a function $g: y \longrightarrow x$. The domain of g is the range of f and the range of g is the domain of f. Finally, since

$$(fg)(y) = f(x) = y$$

and

$$(gf)(x) = g(y) = x,$$

we see that f and g are inverses of each other. Therefore f has an inverse and $f^{-1} = g$.

Definition 4-2. A function f is said to be strictly increasing if its graph is everywhere rising toward the right, if, that is, for any two elements x_1 and x_2 of the domain of f, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

An important corollary of Theorem 4-2 concerns strictly increasing functions.

Corollary 4-2-1. A function f which is strictly increasing has an inverse.

Proof. If x_1 and x_2 are any two elements of the domain of f, then either $x_1 < x_2$, in which case $f(x_1) < f(x_2)$ by hypothesis, or $x_2 < x_1$, in which case $f(x_2) < f(x_1)$. In either case, $f(x_1) \neq f(x_2)$. Hence f is one-to-one and therefore has an inverse by Theorem 4-2.

A similar result holds for strictly decreasing functions; see Exercise 5.

Theorems 4-1 and 4-2 provide an answer to our first question, which was: Under what circumstances does a function have an inverse? We summarize this answer in Theorem 4-3.

Theorem 4-3. A function has an inverse if and only if it is one-to-one.

As we might reasonably expect, there exists a rather simple relationship between the graph of a function f and the graph of its inverse f^{-1}. If, for example, r and s are real numbers such that $r = f(s)$, then $P(s, r)$ is, by definition, a point of the graph of f. But if $r = f(s)$, then

$s = f^{-1}(r)$, and it follows, again by definition, that $Q(r, s)$ is a point of the graph of f^{-1}. Since this argument is quite general, we can conclude that, for each point $P(s, r)$ of the graph of f, there is a point $Q(r, s)$ of the graph of f^{-1}, and conversely; either graph can be changed into the other by merely interchanging the first and second coordinates of each point. To picture the relative positions of P and Q we should plot a few points and contemplate the results. (See Figure 4-9a, in which corresponding points of each pair $P(s, r)$, $Q(r, s)$ have been joined together.)

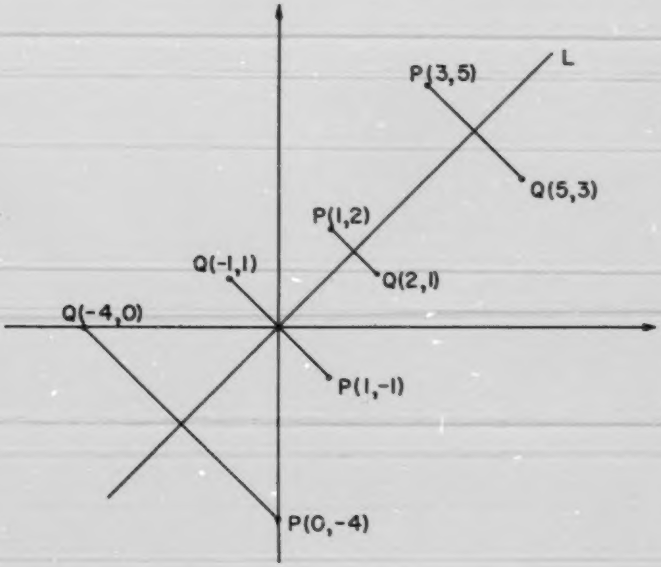


Figure 4-9a

The presence of the line $L = \{(x, y) : y = x\}$ illustrates a striking fact: With respect to the line L, corresponding points are mirror images of each other! Thus we see that the graph of the inverse of a function f is the image of the graph of f in a mirror placed on its edge, perpendicular to the page, along the line L. This fact suggests the following (messy) way to obtain the graph of f^{-1} from that of f. Merely trace the graph of f in ink that dries very slowly, and then fold the paper carefully along the line L. The wet ink will then trace the graph of f^{-1} automatically. (See Figure 4-9b.)

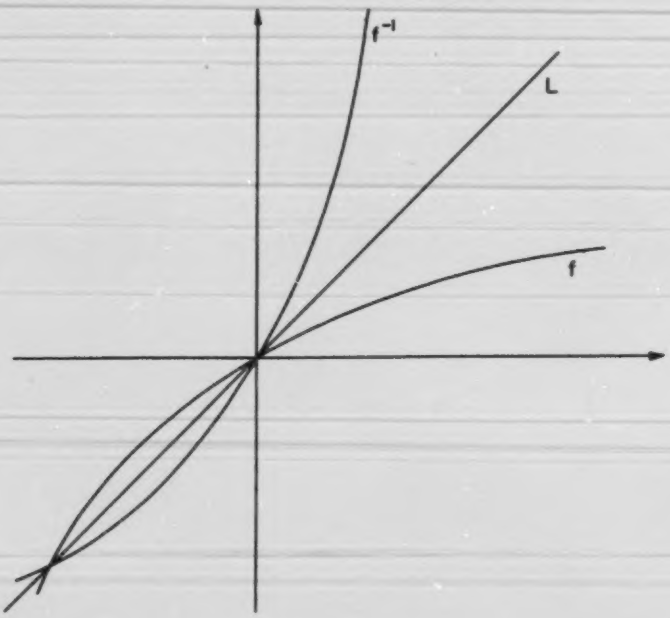


Figure 4-9b.

Exercises 4-9

1. Find the inverse of each of the following functions:
   a) $x \rightarrow 4x - 5$
   b) $x \rightarrow \frac{3}{x} + 8$
   c) $x \rightarrow x^3 - 2$

2. Solve each of the following equations for x in terms of y and compare your answers with those of Exercise 1:
   a) $y = 4x - 5$
   b) $y = \frac{3}{x} + 8$
   c) $y = x^3 - 2$.

3. Justify the following in terms of composite functions, inverse functions, and functions which associate integers with ordered pairs of digits. "A common conjuring trick is to ask a boy among the audience to throw two dice, or to select at random from a box a domino on each half of which is a number. The boy is then told to recollect the two numbers thus obtained, to choose either of them, to multiply it by 5, to add 7 to the result, to double this result, and lastly to add to this the other number. From the number thus obtained, the conjurer subtracts 14, and obtains a number of two digits which are the two numbers chosen originally." (W. W. Rouse Ball)

4. We know that each line parallel to the y-axis meets the graph of a function in at most one point. For what kind of function does each line parallel to the x-axis meet the graph in at most one point?

5. A function f is said to be strictly decreasing if, for any two elements x_1 and x_2 of its domain, $x_1 < x_2$ implies $f(x_1) > f(x_2)$. Prove that every strictly decreasing function has an inverse.

6. a) Sketch a graph of $f: x \longrightarrow x^2$, $x \in R$. Show that f does not have an inverse.
   b) Sketch graphs of $f_1: x \longrightarrow x^2$, $x \geq 0$ and $f_2: x \longrightarrow x^2$, $x < 0$, and determine the inverses of f_1 and f_2.
   c) What relationship exists among the domains of f, f_1, and f_2? (f_1 is called the restriction of f to the domain $\{x : x \geq 0\}$, and f_2 is similarly the restriction of f to the domain $\{x : x < 0\}$.)

7. a) Sketch a graph of $f: x \rightarrow \sqrt{4 - x^2}$ and show that f does not have an inverse.

   b) Divide the domain of f into two parts such that the restriction of f to either part has an inverse.

8. Do Exercise 7 for $f: x \rightarrow x^2 - 4x$.

9. Divide the domain of $f: x \rightarrow x^3 - 3x$ into three parts such that the restriction of f to each has an inverse.

10. If the function $f: x \rightarrow a_0 + a_1x + a_2x^2 + a_3x^3$ has an inverse, what must be the nature of the zeros of the slope function f'?

4-10. Logarithmic Functions.

Thus far we have been concerned with the exponential functions

$$f: x \rightarrow a^x.$$

Hereafter, we assume that $a > 1$. Then f is one-to-one. That is, if we start with any two different values of x we obtain two different values of the function. Because f is one-to-one, f has an inverse function f^{-1}. The graph of f^{-1} is the reflection of the graph of f in the line $y = x$, since (d, c) is a point on the graph of f^{-1} if and only if (c, d) is a point on the graph of f. (See Figure 4-10a.) The domain of f^{-1} is the set of positive real numbers (which is the range of f) and the range of f^{-1} is the set of all real numbers (which is the domain of f). You should verify this from Figure 4-10a.

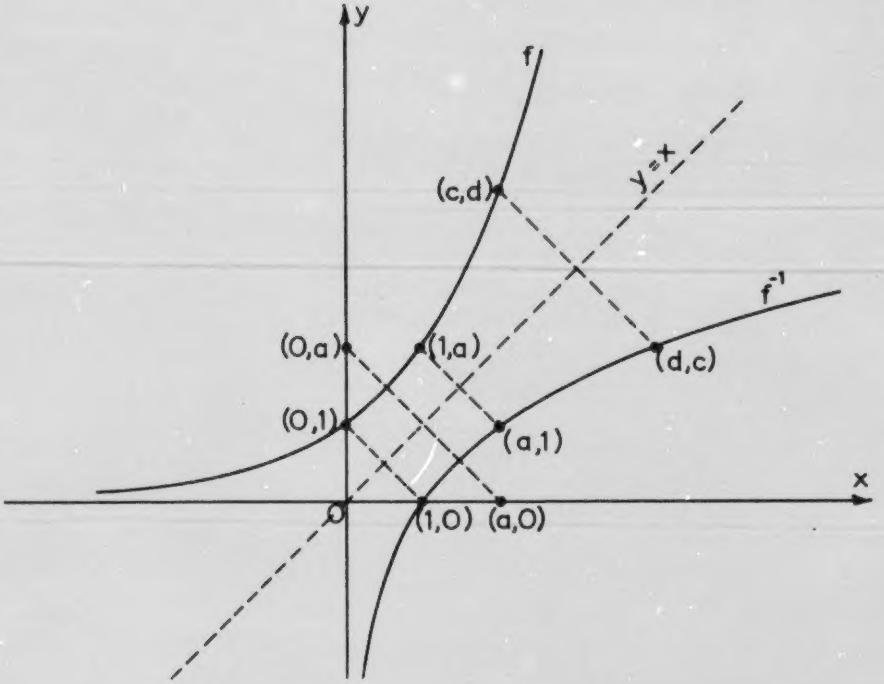


Figure 4-10a. The graphs of f and f^{-1}.

The function f^{-1} is called the logarithm to the base a and is denoted by the symbol $\log_a$. Hence we have

$$f^{-1}: x \rightarrow \log_a x.$$

Inverse Functions

$$\begin{aligned} f&: x \rightarrow a^x \\ f^{-1}&: x \rightarrow \log_a x \end{aligned} \qquad (1)$$

Examples: If $f: x \rightarrow 2^x$, then $f^{-1}: x \rightarrow \log_2 x$.

If $f: x \rightarrow e^x$, then $f^{-1}: x \rightarrow \log_e x$.

A very useful way of thinking about logarithms is derived from the fact that if f is a function and f^{-1} is its inverse, then

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

Hence $f^{-1}(a^x) = x$ and $f(\log_a x) = x$,

or $\log_a(a^x) = x$ and $a^{\log_a x} = x$. (2)

This second identity clearly shows that $\log_a x$ is the exponent of a needed to yield x. Thus

$$y = \log_a x \quad \text{and} \quad a^y = x \qquad (3)$$

are equivalent equations. This accounts for the fact that logarithms are frequently defined as exponents. Thus, for example, since $10^3 = 1000$ and $10^{-2} = 0.01$, we write $\log_{10}1000 = 3$ and $\log_{10}0.01 = -2$; also, since $e^{1/2} \approx 1.6487$ and $e^{-1} \approx 0.3679$, we write $\log_e 1.6487 \approx 1/2$ and $\log_e 0.3679 \approx -1$. Identity (2) enables us to see that

$$10^{\log_{10}1000} = 1000 \qquad e^{\log_e 1.6487} = 1.6487$$

$$10^{\log_{10}0.01} = 0.01 \qquad e^{\log_e 0.3679} = 0.3679\ .$$

On the other hand, suppose we know that $\log_{10}e \approx 0.43429$; using (3) we obtain $10^{0.43429} \approx e$. In other words, the logarithm of e to the base 10 is the exponent of 10 which yields e.

The properties of the function f^{-1} follow immediately from those for the exponential function f as we proceed to show. If f is the function

$$f: x \longrightarrow a^x,$$

the familiar equation

$$a^0 = 1$$

may be written

$$f(0) = 1.$$

Then of course

$$f^{-1}(1) = 0.$$

This is immediately clear from the graph (see Figure 4-10a). In logarithmic notation this result is written

$$\log_a 1 = 0.$$

The fundamental equation

$$a^{x_1 + x_2} = a^{x_1} a^{x_2}$$

becomes in terms of the exponential function f,

$$f(x_1 + x_2) = f(x_1)f(x_2). \qquad (4)$$

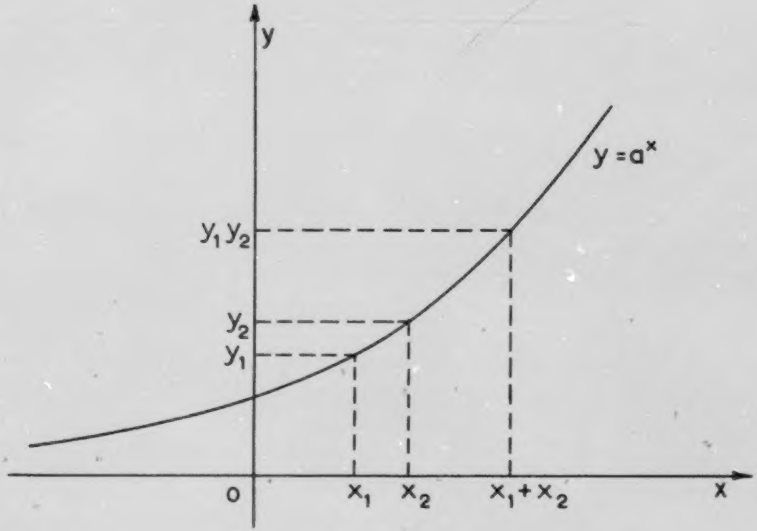


Figure 4-10b.
A fundamental property of exponential functions.

On the graph (Figure 4-10b) of $y = a^x$, the identity (4) means that the ordinate which corresponds to the sum of x_1 and x_2 is the product of the ordinates y_1 and y_2. In other words, addition on the x-axis corresponds to multiplication on the y-axis. Since the x and y axes are interchanged in the reflection of the graph of f in the line $y = x$ to obtain the graph of f^{-1}, we expect that for f^{-1}, multiplication on the x-axis corresponds to addition on the y-axis. Hence for all real positive numbers x_1 and x_2

$$\log_a x_1 x_2 = y_1 + y_2 = \log_a x_1 + \log_a x_2 , \qquad (5)$$

which expresses a familiar property of logarithmic functions. (See Figure 4-10c.)

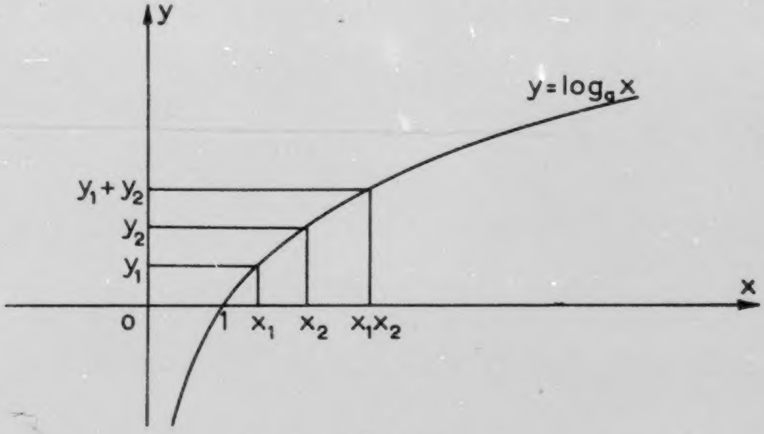


Figure 4-10c.
A fundamental property of logarithmic functions.

Similarly, from the fact that

$$a^{xp} = (a^x)^p , \quad p \text{ rational},$$

or

$$f(xp) = (f(x))^p \qquad (6)$$

we see that for an exponential function f, multiplication of x by p corresponds to raising $y = f(x)$ to the power p. In other words, if $x \rightarrow f(x)$,

$$xp \rightarrow \left[f(x)\right]^p .$$

For the inverse function, raising x to the power p should correspond to multiplying y by p. That is

$$\log_a x^p = p \log_a x \qquad (7)$$

which expresses another well-known property of the logarithmic function.

The results (5) and (7) may be established without appeal to the figure. From Equation (4)

$$f(x_1 + x_2) = f(x_1)f(x_2),$$

hence
$$f^{-1}\left[f(x_1 + x_2)\right] = f^{-1}\left[f(x_1)f(x_2)\right]. \qquad (8)$$

Now $f^{-1}f(x_1 + x_2) = x_1 + x_2$. We set $y_1 = f(x_1)$ so that $x_1 = f^{-1}(y_1)$ and $y_2 = f(x_2)$ so that $x_2 = f^{-1}(y_2)$. Substitution in (8) gives

$$x_1 + x_2 = f^{-1}(y_1y_2)$$

But
$$x_1 + x_2 = f^{-1}(y_1) + f^{-1}(y_2).$$

If we replace f^{-1} by its name $\log_a$ we have for all real positive numbers y_1 and y_2

$$\log_a(y_1y_2) = \log_a y_1 + \log_a y_2$$

which is equivalent to (5).

Similarly, from Equation (6)

$$\left[f(x)\right]^p = f(xp),$$

$$f^{-1}\left[f(x)\right]^p = f^{-1}f(xp)$$
$$= xp. \qquad (9)$$

With $y = f(x)$ and $x = f^{-1}(y)$, Equation (9) becomes

$$f^{-1}\left(y^p\right) = pf^{-1}(y),$$

that is,

$$\log_a y^p = p \log_a y,$$

which is equivalent to (7).

Properties (5) and (7) enable us to use Table 4-6 to obtain logarithms to the base e of numbers not appearing in the table. Thus, to find $\log_e 10$ we shall find $\log_e 2$ and $\log_e 5$: from (5) we know that $\log_e 10 = \log_e 2 + \log_e 5$.

Example 1. Find $\log_e 2$.

Solution. In order to apply (5) we must write 2 as the product of entries in the table,

$$2 \approx (1.9155)(1.0408)(1.003) \quad \text{(See note below.)}$$

From (5) we have

$$\begin{aligned} \log_e 2 &\approx \log_e 1.9155 + \log_e 1.0408 + \log_e 1.003 \\ &\approx 0.65 + 0.04 + 0.003 = 0.693. \end{aligned}$$

Note: Since 1.9155 is the largest entry (in the e^x column of Table 4-6) not greater than 2 we divide: $2 \div 1.9155 \approx 1.0441$. Furthermore, the largest entry not greater than 1.0441 is 1.0408, and $1.0441 \div 1.0408 \approx 1.003$, so that $2 \approx (1.9155)(1.0441) \approx (1.9155)(1.0408)(1.003)$.

Example 2. Find $\log_e 5$.

Solution. We first write 5 as the product of entries in Table 4-6,

$$5 \approx (4.4817)(1.115648) \approx (4.4817)(1.1052)(1.009).$$

From (5)

$$\log_e 5 \approx 1.50 + 0.10 + 0.009 = 1.609.$$

In Exercise 1 you are asked to find $\log_e 10$. We include a graph of $x \rightarrow \log_e x$ in Figure 4-10d for your convenience.

[sec.

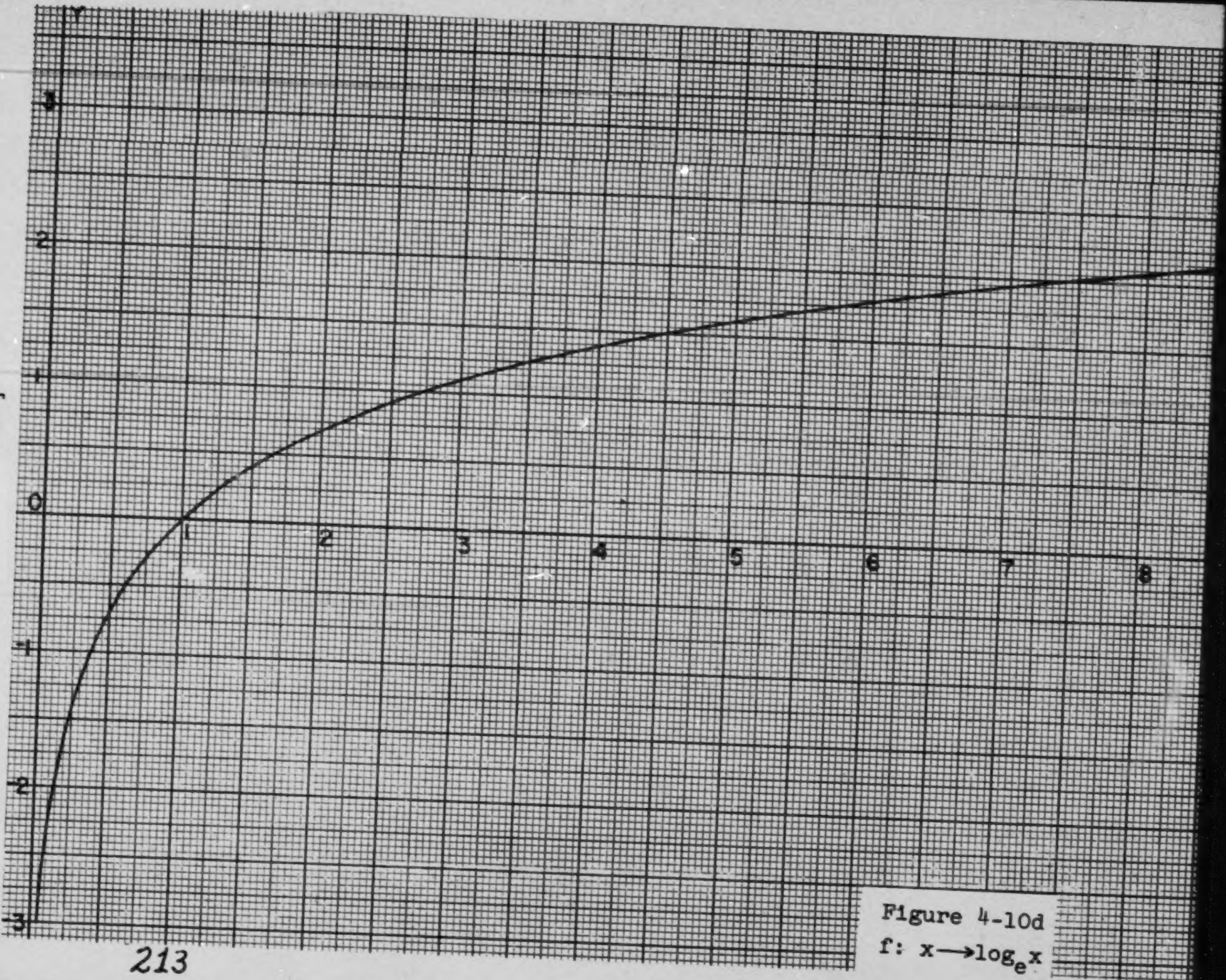


Figure 4-10d

f: $x \longrightarrow \log_e x$

Exercises 4-10

1. Use the results of Examples 1 and 2 to obtain $\log_e 10$.

2. From $1.25 \approx (1.2214)(1.0202)(1.003)$ and the data in Table 4-6 determine $\log_e 5/4$.

3. Use the result of Example 1 to obtain $\log_e 4$.

4. a) Use your results in Exercises 2 and 3 to obtain $\log_e 5$.

   b) Compare your answer with that given in Example 2.

5. From $3 \approx (2.7183)(1.0513)(1.0408)(1.008)$ and the data in Table 4-6 determine $\log_e 3$.

6. Use values for $\log_e 2$, $\log_e 3$, and $\log_e 5$ obtained in this section to determine $\log_e 0.25$, $\log_e 0.5$, $\log_e 2/3$, $\log_e 5/3$, $\log_e 2.5$, $\log_e 6$, $\log_e 8$, $\log_e 9$.

7. Use the graph of $f^{-1}: x \longrightarrow \log_e x$ (Figure 4-10d) to estimate the value of $\log_e 0.25$, $\log_e 0.5$, $\log_e 2/3$, $\log_e 5/3$, $\log_e 2.5$, $\log_e 6$, $\log_e 8$, $\log_e 9$.

8. Compare your results obtained in Exercises 6 and 7.

9. What is the value of x if $32 = 4^x$?

10. If $a \cdot a^m = (a^2)^m$, what is the value of m?

11. Prove that for x any real number > 0, $\log_a(x \cdot \frac{1}{x}) = 0$, and hence $\log_a(\frac{1}{x}) = -\log_a x$.

12. Prove that $\log_a \left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2$.

13. Show that $\log_a a = 1$. Write this equation in exponential form.

14. Express in exponential form

    a) $\log_{10} 35 = y$

    b) $\log_2 25 = x$

    c) $\log_e d = b$

    d) $2 \log_{10} 5 = x$

    e) $\log_e 7 + \log_e 6 = x$

    f) $\frac{1}{2}\log_e 25 - \log_e 2 = x$

15. Given $\log_{10}2 = 0.3010$ find

$\log_{10}5$, $\log_{10}(1/2)$, $\log_{10}(25/4)$, $\log_{10}(128/5)$

16. Express each of the following in logarithmic form.

a) $\sqrt[3]{125} = 5$

b) $10^{-2} = 0.01$

c) $27^{4/3} = 81$

d) $0.04^{3/2} = 0.008$

e) $\sqrt{\sqrt{16}} = 2$

17. Solve for x.

$$\log_6(x + 9) + \log_6 x = 2$$

4-11. Special Bases for Logarithms

Our study of exponential functions was founded on the use 2 as a base. In practice, the bases most generally used are e and 10. Nevertheless, logarithms to the base 2 are of some importance. For example, they play an important role in information theory, a very recently invented mathematical discipline of considerable and growing importance in the design and operation of telephone, radio, radar, and other communication systems.

"In the modern theory of information, originated by communication engineers, the usual unit of quantity of information is the binary digit (abbreviated bit). Thus if a language of signals is to be composed of three binary digits in succession, the language contains eight messages, namely, 000, 001, ... , 111, and each message is said to contain three bits of information. Note that this quantity of information is $\log_2 8$. In general if there are N different messages, the quantity of information in each message is said to be $\log_2 N$." *

---

*M. Richardson, Fundamentals of Mathematics, Rev. Ed., Macmillan, 1958, pp. 172,173.

Logarithms to the base e are usually called natural logarithms; logarithms to the base 10 are called common logarithms. We shall write $\log_e x$ simply as $\ln x$ and show how to express common logarithms in terms of natural logarithms. We first consider the general problem of expressing the logarithm to any base a in terms of the logarithm to any other base b.

By identity (2) of Section 4-10

$$x = b^{\log_b x} . \qquad (1)$$

We take the logarithm of $x = b^{\log_b x}$ to the base a and use equation (7) of Section 4-10 to obtain

$$\log_a x = (\log_b x)(\log_a b). \qquad (2)$$

If we set $x = a$, (2) gives

$$1 = (\log_b a)(\log_a b)$$

or

$$\log_a b = \frac{1}{\log_b a} . \qquad (3)$$

To write common logarithms in terms of natural logarithms we use (2), thus

$$\ln x = (\log_{10} x)(\ln 10),$$

or

$$\log_{10} x = \frac{\ln x}{\ln 10} .$$

For example, $\log_{10} e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10} \approx \frac{1}{2.302} \approx 0.434.$

Similarly, by using the entry 0.30103 to be found in a 5-place table of common logarithms, we have

$$\ln 2 = \frac{\log_{10} 2}{\log_{10} e} \approx \frac{0.301}{0.434} \approx 0.693$$

which agrees with the result obtained in Example 1, Section 4-10. In Section 4 we defined $e = 2^{1/k}$, so that $e^k = 2$; we now note that $k = \ln 2$ since $e^{\ln 2} = 2$.

Exercises 4-11

1. a) If $2^p = 26$, find $\log_2 26$ in terms of p.

   b) If $\log_e x = 5$, find x. (Use Table 4-6.)

   c) Find $\log_3(3^{1/4})$.

   d) Find $\log_2(8 \times 16)$.

2. Express each of the following logarithms in terms of r, s, and t, if $r = \ln 2$, $s = \ln 3$, $t = \ln 5$.

   a) $\ln 4$
   b) $\ln 6$
   c) $\ln 1/8$
   d) $\ln 10$
   e) $\ln 2.5$
   f) $\ln 2/9$
   g) $\ln \frac{5}{9}\sqrt{3}$
   h) $\ln 8\sqrt[3]{100}$

3. Write the following logarithms as numbers.

   a) $\log_{10}1000$
   b) $\log_{0.01}0.001$
   c) $\log_3(1/81)$
   d) $\log_4 32$
   e) $\log_{10}(0.0001)$
   f) $\log_{0.5}16$
   g) $\ln e^3$
   h) $\ln \sqrt{e}$
   i) $\log_{81}27$
   j) $\log_2\sqrt{32}$

4. Given $\ln 10 \approx 2.3026$, $\ln 3 \approx 1.0986$, find

   a) $\log_{10}3$
   b) $\log_{10}e$
   c) $\log_3 10$
   d) $\ln 100$
   e) $\ln 30$
   f) $\ln 300$
   g) $\ln 0.3$
   h) $\ln 0.003$

5. In each case determine the value of x.

   a) $4^{\log_4 5} + 3^{\log_3 5} = 2^{\log_2 x}$

   b) $\log_{10}(x^2 - 1) - 2\log_{10}(x - 1) = \log_{10}3$

   c) $7^{\log_x 5} = 5$

6. Solve the following equations.

a) $\ln x = 0$
b) $\ln x + 1 = 0$
c) $\ln x = 1$
d) $\ln (x - 2) = 3$
e) $\ln x + 3 = 0$
f) $\ln (2x - 1) + 2 = 0$

7. Show that $\left(e^{\ln 10}\right)^{\log_{10}e} = 10^{\log_{10}e} = e$. Hence show that $\ln 10 \log_{10}e = 1$.

8. For what value(s) of x does it hold that

a) $\log_c x = 0$
b) $\log_x x = 1$
c) $x^{\log_x c} = c$
d) $\log_x 2^x = 2$

9. Explain why the number 1 cannot be used as a base for logarithms. (Hint: Examine the exponential function

$$f: x \longrightarrow 1^x$$

to see if it has an inverse function.)

10. Find the values of x for which

$$(\ln x)^2 = \ln x^2 .$$

Note: Use a table of logarithms in Exercises 11 and 12.

11. How long will it take N dollars to double itself at 4 per cent compounded annually? At 3 per cent compounded quarterly?

12. If interest is compounded quarterly, at what rate should N dollars be invested to double in 10 years?

13. A culture of bacteria has a population of 10,000 initially and 60,000 an hour and a half later. Assuming ideal growth conditions, find the time required to get a culture of 500,000 bacteria.

14. a) Through the point (1, 4) draw a line L_1 with slope $m = 2/3$.
    b) Draw the line L_2 which is symmetric to L_1 with respect to the line $y = x$.
    c) What point on L_2 corresponds to the point (1, 4) on L_1?
    d) What is the slope of L_2?
    e) Consider the general case: line L_1 drawn through point (r, s) with slope = m, and line L_2 symmetric to L_1 with respect to the line $y = x$. What point on L_2 corresponds to the point (r, s) on L_1? What is the slope of L_2?

15. a) Plot the points (x, e^x) for which $x = -1.6, -1.4, \ldots, 1.2, 1.4$.
    b) Through each of these points draw the graph of a line having slope $m = e^x$.
    c) Show that these lines suggest the shape of the graph of $f: x \rightarrow e^x$.

16. a) For each point located in Exercise 15(a), locate the corresponding point which is symmetric with respect to the line $y = x$; then through these points draw lines symmetric to those of Exercise 15(b) with respect to the line $y = x$.
    b) Show that each point located in Exercise 16(a) lies on the graph of $x \rightarrow \ln x$.
    c) Compare the slopes of the lines drawn in Exercise 16(a) with those of Exercise 15(b).

17. Using the graphs of $x \rightarrow e^x$ and $x \rightarrow \ln x$ in Figures 4-6b and 4-10d, compare the slopes of the respective graphs at $x = 0$.

4-12. Computation of e^x and $\ln x$.

Because of its simple properties, the most important of the exponential functions is $x \longrightarrow e^x$. Similarly $x \longrightarrow \ln x$ is the most important of the logarithmic functions. The computation of the values of these functions is therefore of great significance in mathematics.

We found approximate values of e^x on the basis of a table of powers of 2^x. We then determined values of the function $x \longrightarrow \ln x$ which is inverse to $x \longrightarrow e^x$.

A mathematician would proceed in a different way. He would compute the values of e^x directly (that is, without any reference to powers of 2). We shall describe the general features of this method. An appendix carries the development somewhat further.

A brief review of our treatment of polynomial functions will be helpful. We notice, for example, that in graphing

$$f: x \longrightarrow 2 + 3x + x^2 - x^3$$

near (0, 2), we could replace $f(x)$ by $g(x) = 2 + 3x$ or by $h(x) = 2 + 3x + x^2$. For $|x|$ small enough, $f(x)$ is approximately equal to $g(x)$,

$$f(x) \approx g(x) .$$

A better approximation is given by $h(x)$,

$$f(x) \approx h(x) .$$

That is, the error made in replacing $f(x)$ by $h(x)$ for a given x near 0, will ordinarily be less than the error made in replacing $f(x)$ by $g(x)$. Of course, if we include the final term $-x^3$, there is no error at all.

Turning to e^x, the idea is this: We try to replace $x \longrightarrow e^x$ by a polynomial function whose values approximate e^x. Let us begin with $|x|$ small, that is values of x which are near zero. Since the graph intersects the y-axis at (0, 1) with slope 1, we have the linear approximation

$$e^x \approx 1 + x \qquad (1)$$

To obtain a better approximation to e^x, we need to use polynomials of higher degree

$$e^x \approx 1 + x + a_2x^2 ,$$

$$e^x \approx 1 + x + a_2x^2 + a_3x^3 ,$$

$$\dots$$

The problem is to specify the values of a_2, a_3, etc. The appendix shows how this may be done. Here we merely report that the required approximations are

$$e^x \approx 1 + x + \frac{x^2}{2} \tag{2}$$

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} \tag{3}$$

$$\dots$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

where $n! = 1\cdot 2\cdot 3\cdot \dots \cdot n$.

In Figure 4-12a, we have drawn graphs of $x \rightarrow e^x$ and of $x \rightarrow 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3}$ (Compare with (3)).

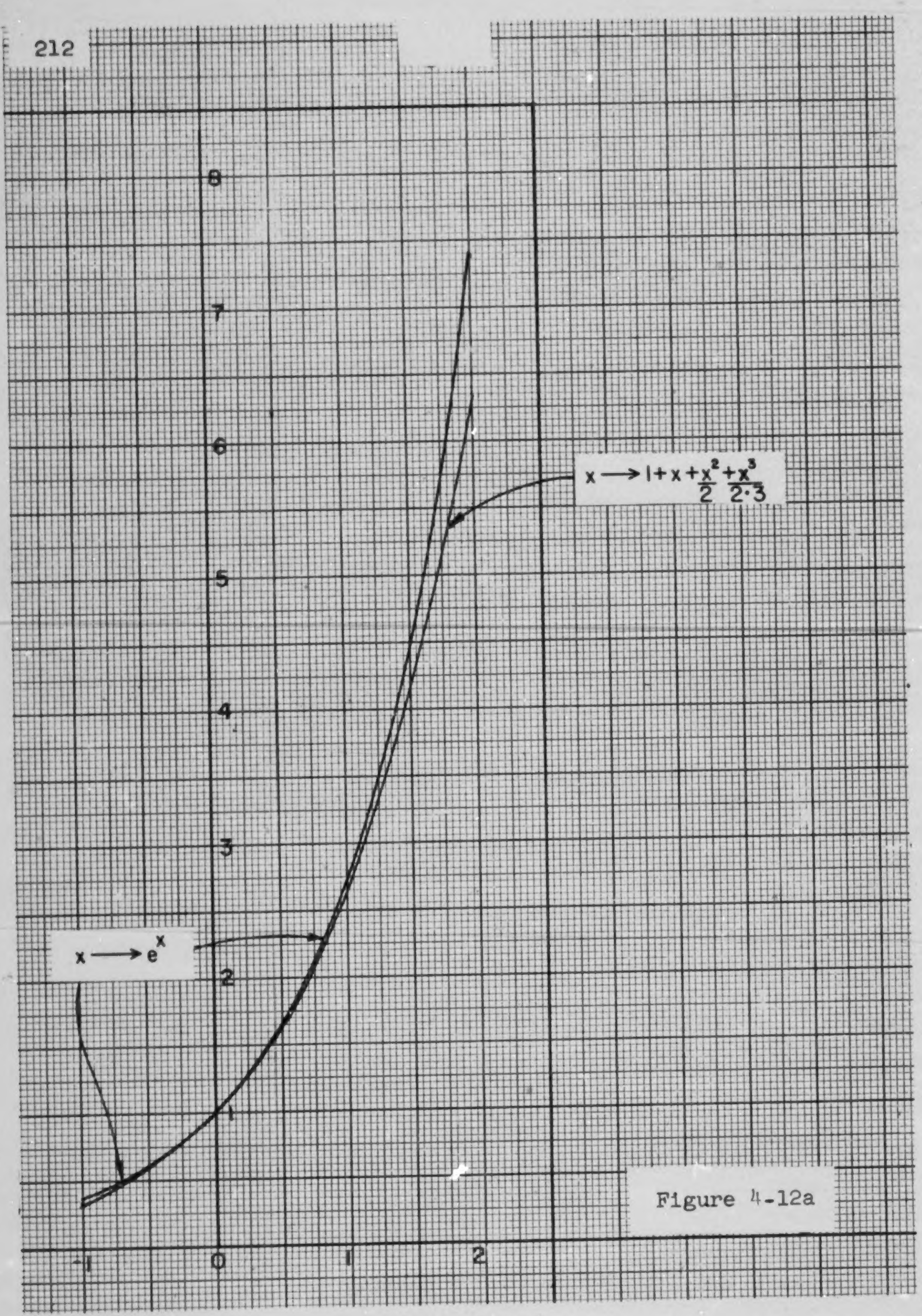


Figure 4-12a

These graphs make it plausible that at least the first approximations (1), (2), (3) listed are correct.

As we know, $x \longrightarrow e^x$ is not a polynomial function. There is therefore a fundamental difference between the problem of approximating e^x by using polynomials and the previous case in which the given function f is a polynomial function. If f is a polynomial function, we can use approximating polynomials of higher and higher degree until we reach a polynomial which is identical with $f(x)$ and which therefore gives exact values for f. For e^x this is impossible. In the case of the exponential, no matter how large an n we choose, the value of the approximating polynomial for a given x ($\neq 0$) will always be in error by some amount. The most that we can hope for is that we can choose n large enough so that for a given x,

$$1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

will differ from e^x by an arbitrarily small amount. Fortunately this hope is realized.

In particular, for $x = 1$ and $n = 10$, we obtain

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{10!}\,. \qquad (4)$$

Rounding off to the eighth decimal place, the right side of (4) has the value 2.71828181. The correct value to eight decimal places is 2.71828183.

We have seen that for the purpose of computation, the exponential function $f: x \longrightarrow e^x$ can be replaced by one of a list of polynomial functions. A similar situation holds for the logarithmic function $f^{-1}: x \longrightarrow \ln x$. However, it is useless to try to approximate $\ln x$ for x near zero since $\ln x$ is not defined at $x = 0$. It is usual to approximate $\ln x$ for x near 1. This may be done by using the following list of polynomial approximations

$$\ln x \approx x - 1 \qquad (5)$$

$$\ln x \approx (x - 1) - \frac{(x - 1)^2}{2} \qquad (6)$$

$$\ln x \approx (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} \qquad (7)$$

For values of x near 1, that is, for $|x-1|$ small, these approximations serve as a satisfactory substitute for $\ln x$. For example:

$$\ln 1.1 \approx .1 \qquad \text{from (5)},$$

$$\ln 1.1 \approx .1 - \frac{.01}{2} = .095 \qquad \text{from (6)},$$

$$\ln 1.1 \approx .1 - \frac{.01}{2} + \frac{.001}{3} = .09533 \qquad \text{from (7)}.$$

The correct value of $\ln 1.1$ to five places is .09531.

Graphs of $x \longrightarrow \ln x$ and of $x \longrightarrow (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ are given in Figure 4-12b. These graphs show that the approximation is good for x near 1, but is poor for x near 0, say at $x = .1$.

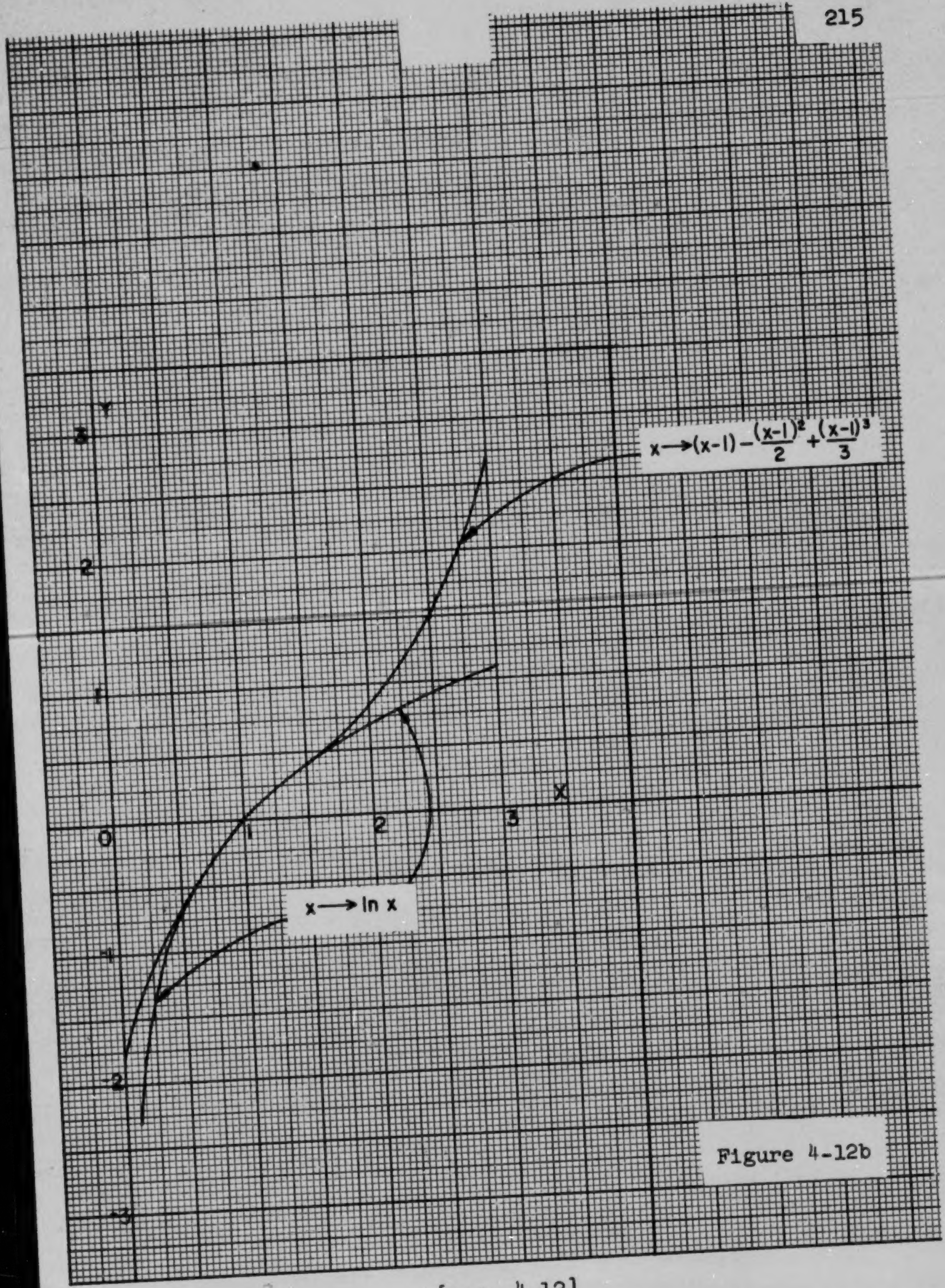

Y
3
2
1
0
1
2
3
X
-1
-2
-3
$x \longrightarrow (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$
$x \longrightarrow \ln x$

Figure 4-12b

This situation is improved by using an approximating polynomial of higher degree. The difficulty is fundamental, however, in the sense that any given polynomial approximation will fail to give satisfactory answers for x near enough to 0. The reason is that as previously stated, $\ln 0$ does not exist, and for x very small, $\ln x$ is negative and numerically very large.

The approximating polynomials are useful for calculating $\ln x$ for $.5 \leq x \leq 1.5$, for example. Other logarithms can be computed from these. Thus, if we know that $\ln 1.4 \approx .33647$, $\ln 1.96 = \ln (1.4)^2 = 2 \ln 1.4 \approx .67294$.

For a further discussion of these matters, the student should consult Appendix 4-18.

Exercises 4-12

1. Use $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ to approximate e^x for $x = .1, .2, .5, -.1, -.2$. Compare the values obtained with those given in Table 4-6.
2. Verify the approximation to e found in (4). Hint: Use the fact that $\frac{1}{5!} = \frac{1}{5} \cdot \frac{1}{4!}$, $\frac{1}{6!} = \frac{1}{6} \cdot \frac{1}{5!}$, etc.
3. According to (5), the tangent to the graph of $x \rightarrow \ln x$ at $(1, 0)$ has the equation $y = x - 1$. Show that this is consistent with the fact that the tangent to the graph of $x \rightarrow e^x$ at $(0, 1)$ has the equation, $y = x + 1$.
4. Use (7) to estimate the value of $\ln 1.2$ and $\ln 1.3$ and compare with the graph of $\ln x$ (Figure 4-10d).
5. From $\ln 1.1 \approx .09531$ find $\ln 1.21$, using one of the properties of logarithms.

4-13. Historical Notes

The theory of logarithms is one of the major achievements of the seventeenth century. However, the rudiments of the notion appeared as early as 1544.

Stifel, who is considered the greatest German algebraist of the 16th century, noticed the advantage in setting up a correspondence between a geometric progression

$$\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32, 64$$

and an arithmetic progression

$$-3, -2, -1, 0, 1, 2, 3, 4, 5, 6.$$

His treatise Arithmetica Integra, published in Latin, might be said to contain the beginning of the theory of exponents and logarithms.

John Napier (1550-1617), the Scotsman, is often regarded as the inventor of logarithms. The object of his study was to facilitate trigonometric calculation. One of the curiosities of mathematical history is the fact that Napier's Table of Logarithms appeared (1614) before exponential symbolism was developed. Although Napier used $1 - 10^{-7} = 0.9999999$ as the basis for his development, the idea of an exponential base does not really apply to Napier's system in which zero is the logarithm of 10^7 and the logarithm increases as the number decreases.

Joost Bürgi (1552-1632) conceived the idea and independently created a table of logarithms of numbers from 10^8 to 10^9 by tens, but did not publish his treatise until 1620. His system was similar to Napier's, but his logarithms increase with the numbers since he selected 1.0001 as his base. It is important to note that Bürgi's object was to simplify all calculations by means of logarithms. In this respect his point of view was broader than Napier's.

Henry Briggs, the Englishman, (1556-1630) was greatly impressed and influenced by Napier's work. He devoted his energies to the construction of a logarithmic table in which zero is the logarithm of 1. An advantage of Briggsian logarithms is that they are built about the base 10. In 1624 Briggs published his work containing the logarithms of numbers from 1 to 20,000 and from 90,000 to 100,000 to 14 places.

Logarithms were developed principally in order to facilitate calculation, and in fact simplified computation to such an extent that they are commonly regarded merely as a labor saving device. This is unfortunate since logarithms are of great value in advanced mathematics, apart from computation.

Brief Bibliography

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4-14. Summary.

The exponential functions are defined by $f: x \rightarrow ka^x$, $a > 0$. This definition requires an interpretation for the symbol a^x. For x a rational exponent, the interpretation $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ is familiar. Meaning is given to a^x for irrational x so that the resulting function is continuous and obeys the laws

$$a^x a^y = a^{x+y}$$

$$(a^x)^r = a^{rx}$$

if x and y are arbitrary real numbers and r is rational.

The graph of $x \rightarrow a^x$ is concave upward everywhere. It has a tangent line at every point. If $a = 2$, the slope of the tangent at $(0, 1)$ is $k \approx 0.693$. The most convenient base is $a = e = 2^{1/k} \approx 2.718$. For this choice of base, the slope is 1 at $(0, 1)$ and e^h at (h, e^h). That is, at every point on the graph of $x \rightarrow e^x$, the slope is equal to the ordinate.

Important phenomena such as growth, radioactive decay and cooling are adequately described by formulas of the type, $y = y_0 e^{\pm cx}$, where y_0 and c are suitable positive constants.

The logarithmic function $f^{-1}: x \rightarrow \log_a x$ is the inverse of $f: x \rightarrow a^x$. Its graph is the reflection of the graph of $x \rightarrow a^x$ in the line $y = x$.

All logarithmic functions have the following properties. If x_1, x_2 and x_3 are any positive numbers

$$\log_a x_1 x_2 = \log_a x_1 + \log_a x_2,$$

$$\log_a \left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2,$$

$$\log_a x_1{}^p = p \log_a x_1, \quad p \text{ rational.}$$

The most important logarithmic functions are $\log_{10}$ and $\log_e = \ln$. To change from base b to base a, we use the equation

$$\log_a x = \log_b x \cdot \log_a b.$$

In particular,

$$\ln x = \log_{10} x \ \ln 10 \approx 2.302 \ \log_{10} x.$$

Tables of e^x and $\ln x$ may be computed from the polynomial approximations

$$e^x \approx 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

and $\ln x \approx (x - 1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dots \pm \dfrac{(x-1)^n}{n}$.

Some discussion of these approximations and the errors associated with using them, is given in the appendices.

Miscellaneous Exercises

1. Assuming that the number N of bacteria at the end of n days is given by $N = Ae^n$, find:
   a) The number of days it takes to double the number of bacteria; express this in terms of the constant k;
   b) The per cent increase per day in the number of bacteria.

2. If f is an exponential function $x \rightarrow ca^x$ such that $f(0) = 2$ and $f(1.5) = 54$, find a and c.

3. Given the function $f: x \rightarrow a^x$ for which $f(2) = 0.25$, find $f(5)$.

4. The gas in an engine expands from a pressure p_1 and volume v_1 to a pressure p_2 and volume v_2 according to the equation

$$\frac{p_1}{p_2} = \left(\frac{v_1}{v_2}\right)^n .$$

(Boyle's law is the special case where $n = 1$.) Solve for v_1 in terms of v_2, p_1, p_2 and n.

5. If $s = \frac{a - ar^n}{1 - r}$, express n in terms of s, a, r.

6. Solve the following equations for x.
   a) $3^{\ln x^3} = 3$ b) $\ln x^2 - 2 \ln\sqrt{x} = 1$

7. If $a^{0.3} = x$ find $\log_x a$.

8. Combine each of the following expressions into a single term.
   a) $\ln \frac{3}{5} + \ln 100 - \ln 12$.
   b) $2 \ln x - \frac{1}{4} \ln y + \frac{2}{3} \ln y - \frac{1}{2} \ln x$.

9. Without graphing, describe the relationship between the graphs of $x \rightarrow ce^x$ and $x \rightarrow e^{cx}$ for $c = -1$.

10. If $f: x \rightarrow 2^x$ and $g: x \rightarrow 3^x$, find
    a) $(fg)(2)$
    b) $(gf)(2)$

11. Given the functions

$$f: x \rightarrow 2^x + 2^{-x}$$

$$g: x \rightarrow 2^x - 2^{-x}$$

find
    a) $f(x) + g(x)$
    b) $f(x) \cdot g(x)$
    c) $[f(x)]^2 - [g(x)]^2$

12. Solve the following equation for x.

$$\log_2(6x + 5) + \log_2 x = 2$$

13. Solve the equation $2^{2x+2} = 9(2^x) - 2$ for x.
(Hint: Let $2^x = y$.)

14. Solve for x: $2^{2x+2} + 2^{x+2} = 3$.

15. Show that no real number x can be found such that

$$\ln (x - 4) - \ln (x + 1) = \ln 6.$$

16. Prove the following special case of the "chain rule" for logarithms:

$$(\log_a b)(\log_b c)(\log_c d) = \log_a d.$$

17. Solve the equation

$$\ln (1 - x) - \ln (1 + x) = 1.$$

18. Sketch the graphs of
    a) $y = \ln |x|$ b) $y = |\ln x|$ c) $y = \ln e^x$

19. At the instantaneous rate of 5 per cent per annum, what will be the value of \$100 at the end of 5 years? (Give your answer correct to the nearest dollar.)

20. What amount must have been deposited 5 years ago to amount to \$100 now at the rate of 5 per cent compounded continuously? (Give your answer correct to the nearest dollar.)

21. At what rate of interest (compounded annually) must we invest \$100 if we want it to double in 10 years? (Give your answer correct to the nearest tenth of one per cent.)

22. Under normal conditions population changes at a rate which is (considered to be) proportional to the population at any time. The population at any time x is then satisfactorily given by

$$N(x) = N_0 e^{kx},$$

where N_0 is the population at time $x = 0$, and k is a suitable constant. If a town had a population of 25,000 in 1950 and 30,000 in 1955, what population is expected in 1965?

23. The function $f: x \longrightarrow e^{-t/RC}$, where t represents time, and R and C are constants, is important in the theory of certain types of electrical circuits. Using Table 4-6, evaluate $f(t)$, given the following data:

a) $t = 1.0, \quad R = 2.0, \quad C = 0.05$

b) $t = 12 \times 10^{-4}, \quad R = 48, \quad C = 25 \times 10^{-6}$

24. Solve the equation $A = e^{-t/RC}$ for t. Using this result, find the value of t for each of the following sets of data. Suggestion: use the graph of $x \longrightarrow \ln x$, (Figure 4-10d).

a) $R = 10, \quad C = 10^{-4}, \quad A = 1.0$

b) $R = 25 \times 10^3, \quad C = 6.0 \times 10^{-4}, \quad A = 0.50$

25. Find, correct to two decimal places, the root of $e^x - x^3 + 3x = 0$ that is nearest 0. (Hint: approximate the root graphically, then use Newton's method.)

26. a) Sketch the graphs of the functions

$$x \longrightarrow 2^x \quad \text{and} \quad x \longrightarrow 2x^3 - 3x^2 - 12x + 1$$

using the same coordinate axes.

b) In how many points do the curves intersect in the interval $-2 \leq x \leq 4$?

c) How many solutions has the equation

$$2^x = 2x^3 - 3x^2 - 12x + 1 \quad \text{if} \quad -2 \leq x \leq 4?$$

d) Answer questions (b) and (c) for the extended interval

$$-2 \leq x \leq 12.$$

27. If a flexible chain or cable is suspended between supports and allowed to hang of its own weight, the curve formed is a catenary. Its equation is

$$y = \frac{a}{2}(e^{x/a} + e^{-x/a}).$$

a) Let $a = 1$. Prepare a table and graph the catenary over the interval $-4 \leq x \leq 4$. (Because of our choice of a, the catenary in this case will be a narrow curve if equal scales are used on the two axes. To offset this effect, choose appropriate scales.)

b) You will notice that the catenary looks somewhat like a parabola. The point $(0, 1)$ is on the graph of the catenary, and the points $(3, 10)$ and $(-3, 10)$ are very close to it. Find the equation of the parabola which passes through these three points, and draw its graph on the same axes as used in part (a).

c) Another approximation to the catenary is given by the equation $y = 1 + x^2/2$, and a better approximation is given by $y = 1 + x^2/2 + x^4/24$. Draw the graphs of these equations on the same axes as used in parts (a) and (b).

28. In the study of probability theory, the normal distribution curve (sometimes described as a "bell-shaped" curve) is of great importance. The equation of this curve is

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \approx 0.4e^{-x^2/2}.$$

a) Prepare a table and graph this curve over the interval $-3 \leq x \leq 3$. (Notice the symmetry of the curve.)

b) By counting squares and multiplying this number by the area of one square (in terms of the scales chosen for the axes), show that the area between the curve and the x-axis, over the interval $-3 \leq x \leq 3$, is very nearly 1.

c) An interesting application of this curve can be made to the scores reported for the College Entrance Examination Board tests. If we associate test scores with values of x according to the following table, then by comparing areas under the curve, we can find what per cent of all scores will fall between any two scores.

x	-3	-2	-1	0	1	2	3
Test score	200	300	400	500	600	700	800

For example, to find what per cent of the scores on a certain test will fall between 500 and 600, we find the area under the curve from $x = 0$ to $x = 1$. By counting squares, you should find that this area is about 0.34, or 34% of the total area under the curve. This means that about 34% of all test scores on any given test will fall between 500 and 600.

Using this technique, find the per cent of test scores that lie between 200 and 300, 300 and 400, ..., 700 and 800. Does the sum of these per cents equal 100%?

Chapter 5

CIRCULAR FUNCTIONS

5-1. Circular Motions and Periodicity

Introduction. From your earliest years you have been aware of motion and of change in the world around you. The rolling of a marble along a crack in the sidewalk, the flight of a ball tossed by a boy at play, the irregular rise and fall of a piece of paper fluttering in the breeze, the zig-zag course of a fish swimming erratically in a tank of water are a few of the varied patterns of movement you can observe. Very often, however, the motions you see have a quality not shared by the few just mentioned. The succession of day and night, the changing of the seasons, the rise and fall of the tides, the circulation of blood through your heart, the passage of the second hand on your watch over the 6 o'clock mark are patterns each having the characteristic quality that the motion involved repeats itself over and over at a regular interval. The measure of this interval is called the period of the motion, while the motion itself is called periodic.

The simplest periodic motion is that of a wheel rotating on its axle. Each complete turn of the wheel brings it back to the position it held at the beginning. After a point of the wheel traverses a certain distance in its path about the axle, it returns to its initial position and retraces its course again. The distance traversed by the point in a complete cycle of its motion is again a period, a period measured in units of length instead of units of time. If it should happen that equal lengths are traversed in equal times, the motion becomes periodic in time as well and the wheel can be used as a clock.*

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*The concept of time itself is inextricably tied up with that of clock, a periodic device which measures off the intervals. It would seem then that periodicity lies at the deepest roots of our understanding of the natural universe. How one decides that a repetitive event recurs at equal intervals of time and can therefore be considered a clock is a profound and difficult problem in the philosophy of physics and does not concern us here. (See Physics, Vol. 1, pp. 9-17, Physical Science Study Committee, Cambridge, Massachusetts, 1957.)

The mathematical analysis of periodic phenomena is a vast and growing field, yet even in the most far-flung applications of the subject, such phenomena are analyzed essentially in terms of the simple periodicity of the path of a point describing a circle. In the treatment of the most intricate of periodicities, wheel motions always lie under the surface. An extended development of the theory of periodic phenomena is far beyond the scope of this course, but the study of the fundamental circular periodicities is certainly within our reach.

Circular Motions. Let us consider first the mathematical aspects of the motion of a point P on a circle. For convenience we take the circle $u^2 + v^2 = 1$, which has its center at the origin of the uv-plane, radius 1 and consequently circumference 2π. Now we consider a moving point P which starts at the point $(1, 0)$ on the u-axis and proceeds in a counterclockwise direction around the circle. We can locate P exactly by knowing the distance x which it has traveled along the circle from $(1, 0)$. The distance x is the length of an arc of the circle. Since every point on the circle $u^2 + v^2 = 1$ has associated with it an ordered pair of real numbers (u, v) as coordinates, we may say that the motion of the point P defines a function $\wp$. With each nonnegative arc length x, we associate an ordered pair of real numbers (u, v), the coordinates of P (Figure 5-1a), that is,

$$\wp: x \longrightarrow (u, v).$$

However, it is inconvenient to work with a function whose range is a set of ordered pairs rather than single numbers. We shall instead define *two* functions as follows:

$$\cos: x \longrightarrow u, \text{ where } u \text{ is the first component of } \wp(x);$$
$$\sin: x \longrightarrow v, \text{ where } v \text{ is the second component of } \wp(x).$$

The terms cos and sin are abbreviations for cosine and sine. It is customary to omit parentheses in writing $\cos(x)$ and $\sin(x)$ and write simply $\cos x$ and $\sin x$. For instance,

$$\wp(0) = (1, 0) \quad : \quad \cos 0 = 1, \ \sin 0 = 0$$
$$\wp(\tfrac{\pi}{2}) = (0, 1) \quad : \quad \cos \tfrac{\pi}{2} = 0, \ \sin \tfrac{\pi}{2} = 1$$

$$\mathcal{P}(\pi) = (-1, 0) \qquad : \qquad \cos \pi = -1, \ \sin \pi = 0$$

$$\mathcal{P}\left(\frac{3\pi}{2}\right) = ? \qquad : \qquad \cos \frac{3\pi}{2} = ?, \ \sin \frac{3\pi}{2} = ?$$

(You should supply the proper symbols in place of the question marks.) From their mode of definition, the sine and cosine are called circular functions. These circular functions are related to but not identical with the familiar functions of angles studied in elementary trigonometry. We shall discuss the difference in Section 5-3, but we should notice now that when we write sin 2, the 2 represents the real number 2 which can be thought of as the measure of the length of a circular arc and not 2 degrees.

$$\mathcal{P} : x \rightarrow (u, v).$$

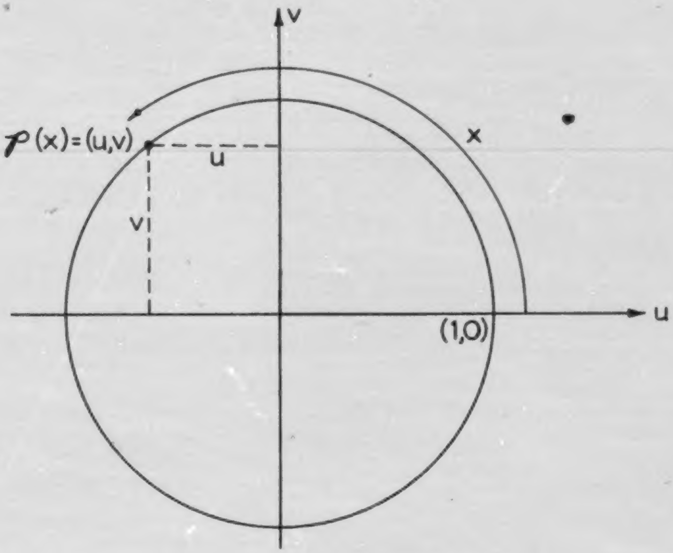

Figure 5-1a. The function $\mathcal{P}$.

Periodicity. From the definition of $\wp$, it follows that $\wp(x) = \wp(x + 2\pi)$ and consequently, $\cos x = \cos(x + 2\pi)$ and $\sin x = \sin(x + 2\pi)$. Functions which have this property of repeating themselves at equal intervals are said to be periodic. More generally, the function f is said to be periodic with period a, $a \neq 0$, if, for all x in the domain of f, $x + a$ is also in the domain and

$$f(x) = f(x + a). \qquad (1)$$

We usually consider the period of such a function as the smallest positive value of a for which (1) is true. The smallest positive period is sometimes called the fundamental period. From this definition we note that each successive addition or subtraction of a brings us back to $f(x)$ again. We may show this by first considering $f(x + 2a)$ where $a > 0$. We have

$$\begin{aligned} f(x + 2a) &= f([x + a] + a) \\ &= f(x + a) \\ &= f(x), \end{aligned}$$

and further

$$\begin{aligned} f(x + 3a) &= f([x + 2a] + a) \\ &= f(x + 2a) \\ &= f(x). \end{aligned}$$

In general, we have

$$f(x + na) = f(x) \quad \text{where } n = 1, 2, 3, \ldots .$$

To show that this holds for negative n, we note that

$$\begin{aligned} f(x - a) &= f([x - a] + a) \\ &= f(x) \\ f(x - 2a) &= f([x - 2a] + a) \\ &= f(x - a) \\ &= f(x). \end{aligned}$$

In general

$$f(x + na) = f(x) \quad \text{where } n = -1, -2, -3 \ldots .$$

We may express these two ideas by

$$f(x + na) = f(x) \quad \text{where } a > 0 \text{ and } n \text{ is any integer.} \tag{2}$$

In other words, to determine all values of f, we need only know its values on the interval $0 \leq x < a$. Thus, suppose the period of f is $a = 2$ so that for all x in the domain of f

$$f(x + 2) = f(x).$$

Then to find $f(7.3)$ we write

$$\begin{aligned} f(7.3) &= f(1.3 + 3 \times 2) \\ &= f(1.3). \end{aligned}$$

To find $f(-7.3)$, we write

$$\begin{aligned} f(-7.3) &= f(0.7 - 4 \times 2) \\ &= f(0.7) \end{aligned}$$

Now returning to the unit circle, we observe that the functions cos and sin behave in exactly this way. From any point P on the circle, a further movement of 2π units around the circle ($a = 2\pi$ in Equation (2)) will return us to P again. Thus the circular functions are periodic with period 2π, and consequently

$$\begin{aligned} \cos(x + 2n\pi) &= \cos x \\ \sin(x + 2n\pi) &= \sin x \end{aligned} \tag{3}$$

where n is any integer. To give meaning to these formulas for negative n, we interpret any clockwise movement on the circle as negative.

So now if we can determine values of cos and sin for $0 \leq x < 2\pi$, we shall have determined their values for all real x.

Exercises 5-1

1. Give five examples of periodic motion, and specify an approximate period for each. (For instance, the rotation of the earth about its own axis is periodic with period 24 hours.)

2. If $\mathcal{p}(x + 2n\pi) = \mathcal{p}(x)$, express each of the following as $\mathcal{p}(b)$, where $0 \leq b < 2\pi$. (For example,
$$\mathcal{p}(\tfrac{5\pi}{2}) = \mathcal{p}(\tfrac{\pi}{2} + 2\pi) = \mathcal{p}(\tfrac{\pi}{2}).)$$

   a) $\mathcal{p}(-\frac{\pi}{2})$ c) $\mathcal{p}(-\frac{3\pi}{2})$

   b) $\mathcal{p}(3\pi)$ d) $\mathcal{p}(4076\pi)$

3. Give the coordinates of $\mathcal{p}(x)$ for each part of Exercise 2 above.

4. Given that $\mathcal{p}$ has the period 2π, find two values of x where $0 \leq x < 4\pi$, such that

   a) $\mathcal{p}(-\frac{\pi}{2}) = \mathcal{p}(x)$, c) $\mathcal{p}(12\pi) = \mathcal{p}(x)$,

   b) $\mathcal{p}(13\pi) = \mathcal{p}(x)$, d) $\mathcal{p}(-\pi) = \mathcal{p}(x)$.

5. For what values of x, where $0 \leq x < 2\pi$, do the following relations hold?

   a) $\cos x = \sin x$,

   b) $\cos x = -\sin x$.

   Hint: Use the fact that $(\cos x, \sin x)$ represents a point on the unit circle.

*6. We know that the functions represented by $\cos x$ and $\sin x$ have period 2π. Find the period of the functions represented by

   a) $\sin 2x$, c) $\cos 4x$,

   b) $\sin \frac{1}{2}x$, d) $\cos \frac{1}{2}x$.

*7. Let f and g be two functions with the same period a. Prove that:

   a) $f + g$ has a period a (not necessarily the fundamental period);

   b) $f \cdot g$ has a period a.

*8. Let f be a function with period a. Prove that the composition gf also has period a for any meaningful choice of a.

*9. Show that the functions sine and cosine have no positive period less than 2π.

5-2. Graphs of Sine and Cosine

We wish now to picture the behavior of the two functions

$$\cos: x \rightarrow u = \cos x$$

$$\sin: x \rightarrow v = \sin x$$

for all real values of x. To do this we shall first look at some of the general properties of these functions, find some specific values of the functions at given values of x, and finally construct their graphs.

We already know that the sine and cosine functions are periodic with period 2π, and so we may restrict our attention to values of x where $0 \leq x < 2\pi$. Now by noting that u and v are the coordinates of a point on a unit circle, we have

$$u^2 + v^2 = 1. \tag{1}$$

But since $u = \cos x$ and $v = \sin x$, we have

$$\cos^2 x + \sin^2 x = 1. \tag{2}$$

If we write (2) as

$$\sin^2 x = 1 - \cos^2 x$$

and as

$$\cos^2 x = 1 - \sin^2 x$$

it is apparent that neither sin x nor cos x can exceed 1 in absolute value, that is,

$$-1 \leq \sin x \leq 1$$

$$-1 \leq \cos x \leq 1.$$

Another property of sin and cos derives from the symmetry of the circle with respect to the u-axis. Two symmetric points on the circle are obtained by proceeding the distance x in both the clockwise and the counterclockwise senses along the circle. In other words, if $\wp(x) = (u, v)$, then

$\wp(-x) = (u, -v)$ (Figure 5-2a). From this we obtain the important symmetric properties

$$\begin{aligned} \cos(-x) &= \cos x \\ \sin(-x) &= -\sin x. \end{aligned} \tag{3}$$

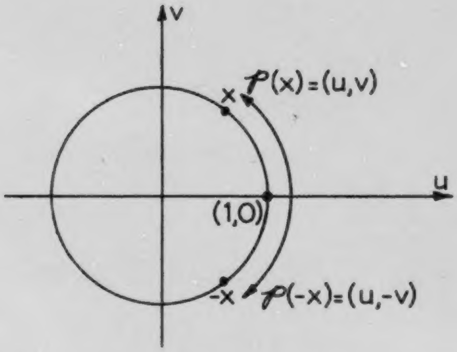

Figure 5-2a. Symmetry relations.

Since we are ultimately interested in graphing $y = \sin x$ and $y = \cos x$, we have managed to narrow our attention to a rectangle of length 2π and of altitude 2 in the xy- plane* as in Figure 5-2b. If we can picture the graph of the functions

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*Since we shall have occasion to refer to two coordinate planes for points (u, v) and (x, y), we wish to point out the distinction between them. The uv-plane contains the unit circle with which we are dealing. This is the circle onto which the function $\wp$ maps the real number x as an arc length. The xy-plane is the plane in which we take the x-axis as the real number line and examine not the point function $\wp(x)$ but the functions $\cos: x \rightarrow y = \cos x$ and $\sin: x \rightarrow y = \sin x$, each of which maps the real number x into another real number.

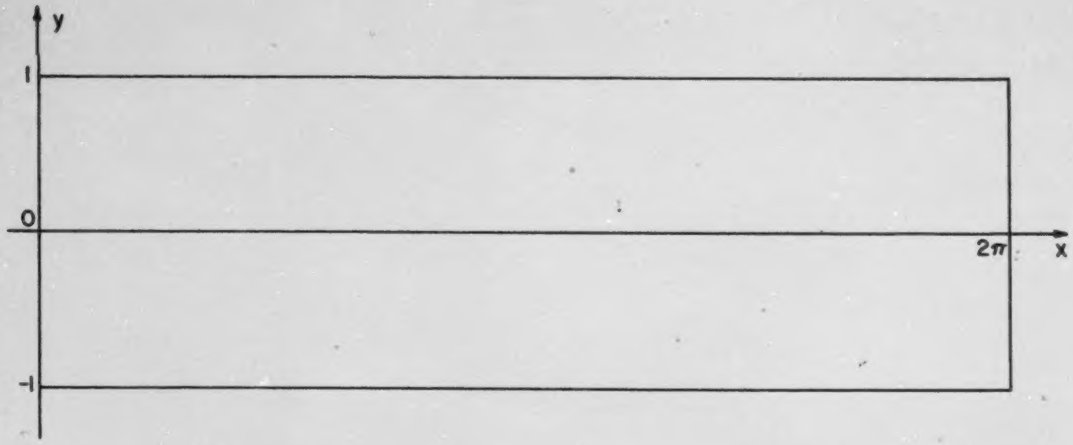

Figure 5-2b. Rectangle to include one cycle of sin or cos.

in the interval $0 \le x < 2\pi$, the periodicity properties of cos and sin will permit us to extend the graph as far as we like by placing the rectangles end to end along the x-axis as in Figure 5-2c.

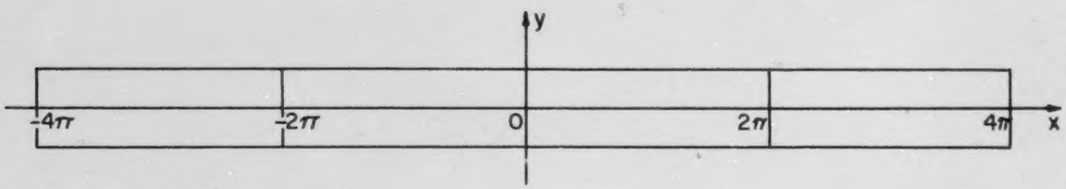

Figure 5-2c. Rectangles of periodicity.

We therefore direct our attention to values of x such that $0 \leq x < 2\pi$. To begin with, the unit circle in the uv-plane is divided into four equal arcs by the axes; each arc is of length $\pi/2$, and the division points correspond to lengths of $x = 0, \pi/2, \pi, 3\pi/2$, with central angles of 0°, 90°, 180°, and 270°, respectively. The corresponding points on the circle will be $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$, as in Figure 5-2d. Since $\cos x = u$ and $\sin x = v$, we have

$$\cos 0 = 1, \qquad \sin 0 = 0,$$

$$\cos \frac{\pi}{2} = 0, \qquad \sin \frac{\pi}{2} = 1,$$

$$\cos \pi = -1, \qquad \sin \pi = 0,$$

$$\cos \frac{3\pi}{2} = 0, \qquad \sin \frac{3\pi}{2} = -1.$$

We next consider the midpoint of each of the quarter circles in Figure 5-2d. These correspond to arc lengths of $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$, with central angles 45°, 135°, 225°, 315°.

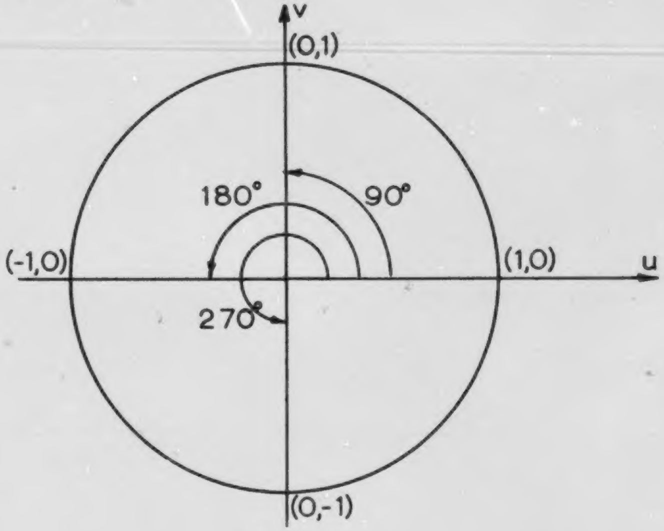

Figure 5-2d. $\mathcal{P}(x)$ for $x = 0, \pi/2, \pi, 3\pi/2$.

If we drop perpendiculars to the u-axis from these points as in Figure 5-2e, we note that radii to the points form angles of 45° with the u-axis. From geometry we know that for a 45° right triangle with hypotenuse 1, the sides are of length $\sqrt{2}/2$ and hence that the coordinates of the midpoints of the quarter circles are $(\sqrt{2}/2, \sqrt{2}/2)$, $(-\sqrt{2}/2, \sqrt{2}/2)$, $(-\sqrt{2}/2, -\sqrt{2}/2)$, and $(\sqrt{2}/2, -\sqrt{2}/2)$, respectively. We may therefore add the following to our list of values:

$\cos \pi/4 = \sqrt{2}/2$	$\sin \pi/4 = \sqrt{2}/2$
$\cos 3\pi/4 = -\sqrt{2}/2$	$\sin 3\pi/4 = \sqrt{2}/2$
$\cos 5\pi/4 = -\sqrt{2}/2$	$\sin 5\pi/4 = -\sqrt{2}/2$
$\cos 7\pi/4 = \sqrt{2}/2$	$\sin 7\pi/4 = -\sqrt{2}/2$

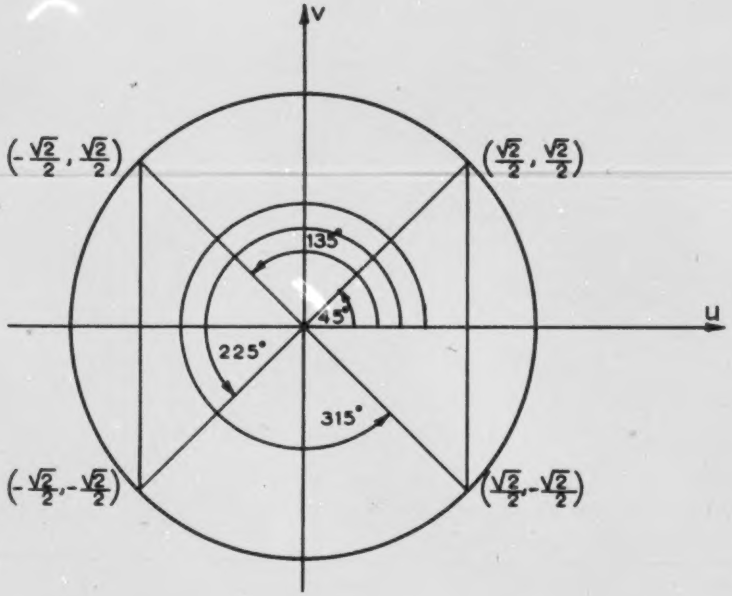
v
u
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
135°
45°
225°
315°
$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

Figure 5-2e. $p(x)$ for $\pi/4$, $3\pi/4$, $5\pi/4$. $7\pi/4$.

We can find the coordinates of the trisection points of the quarter circles by a similar method. In Figure 5-2f, we show only two of the triangles, but the procedure is essentially the same in each quadrant. From the properties of the 30°-60° right triangle, we note that P_1 and P_2 have coordinates

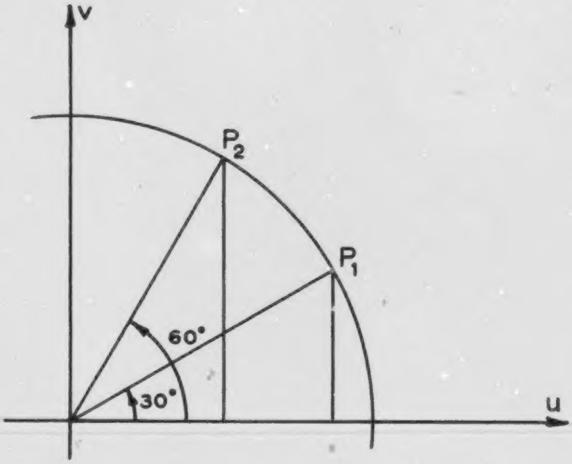

Figure 5-2f. $\wp(x)$ for $x = \pi/6, \pi/3$.

$(\sqrt{3}/2, 1/2)$ and $(1/2, \sqrt{3}/2)$, respectively. We may fill in the coordinates of all of these points of trisection as in Figure 5-2g, from which we can find eight new values for cos and sin. Collecting in one table all of the values which we have so far determined, we have Table 5-1.

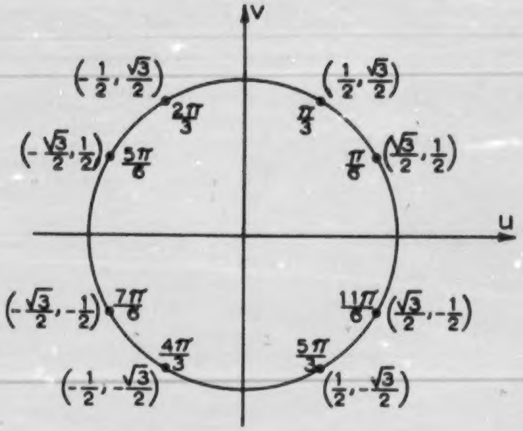
v
u
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{2\pi}{3}$
$\frac{\pi}{3}$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{5\pi}{6}$
$\frac{\pi}{6}$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\frac{7\pi}{6}$
$\frac{11\pi}{6}$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Figure 5-2g. Further values of $\mathcal{P}(x)$.

Table 5-1

Values for cos and sin for one period.

x	cos x	sin x
0	1	0
$\pi/6$	$\sqrt{3}/2 \approx .87$	$1/2$
$\pi/4$	$\sqrt{2}/2 \approx .71$	$\sqrt{2}/2 \approx .71$
$\pi/3$	$1/2$	$\sqrt{3}/2 \approx .87$
$\pi/2$	0	1
$2\pi/3$	$-1/2$	$\sqrt{3}/2 \approx .87$
$3\pi/4$	$-\sqrt{2}/2 \approx -.71$	$\sqrt{2}/2 \approx .71$
$5\pi/6$	$-\sqrt{3}/2 \approx -.87$	$1/2$
π	-1	0
$7\pi/6$	$-\sqrt{3}/2 \approx -.87$	$-1/2$
$5\pi/4$	$-\sqrt{2}/2 \approx -.71$	$-\sqrt{2}/2 \approx -.71$
$4\pi/3$	$-1/2$	$-\sqrt{3}/2 \approx -.87$
$3\pi/2$	0	-1
$5\pi/3$	$1/2$	$-\sqrt{3}/2 \approx -.87$
$7\pi/4$	$\sqrt{2}/2 \approx .71$	$-\sqrt{2}/2 \approx -.71$
$11\pi/6$	$\sqrt{3}/2 \approx .87$	$-1/2$
2π	1	0

With this table we are now in a position to begin graphing sin and cos. Because we wish to look at the graph of these functions over the real numbers, we shall use an xy-plane as usual and work with the points (x, y) where y = cos x or y = sin x. We shall deal separately with each function, taking first y = cos x. From Table 5-1 we can now plot some points in the rectangle in Figure 5-2b, obtaining Figure 5-2h.

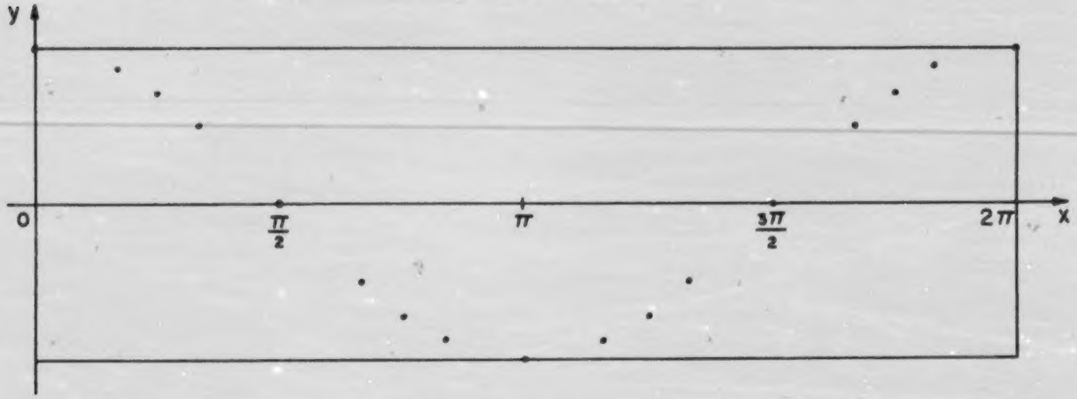

Figure 5-2h. Values of cos: $x \to \cos x$.

By connecting these points by a smooth curve we should obtain a reasonable picture of the function

$$\cos: x \to \cos x$$

as in Figure 5-2i.

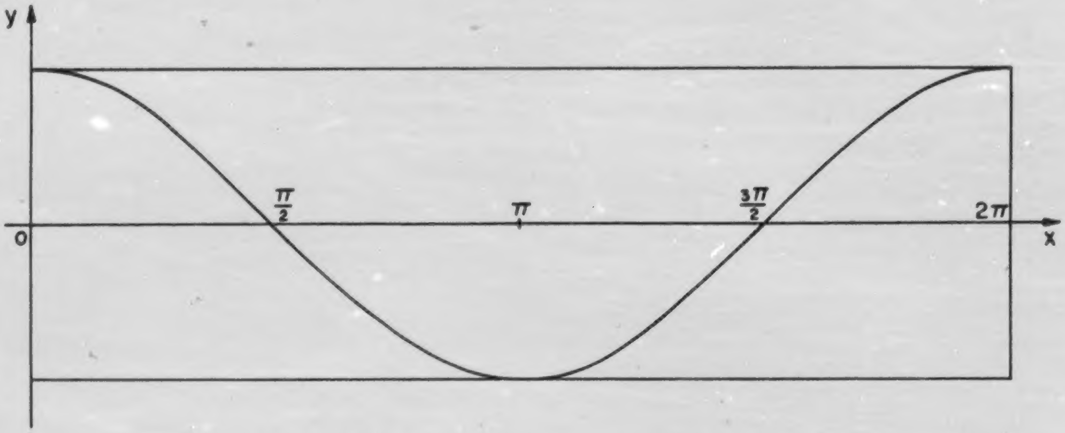

Figure 5-2i. Graph of one cycle of cos.

If we wish to extend our picture to the right and left, we use the periodicity property to obtain Figure 5-2j.

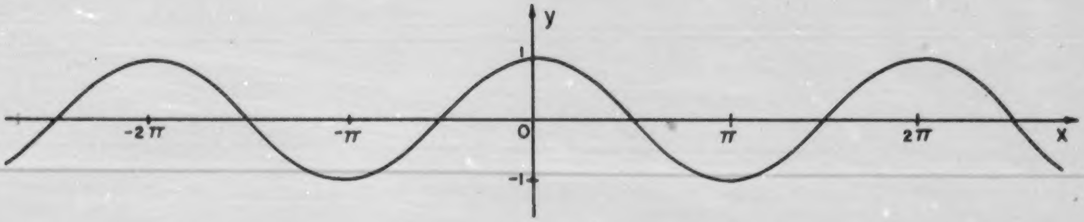

Figure 5-2j. Graph of cos.

A similar treatment of $y = \sin x$ leads to Figures 5-2k, 5-2l, and 5-2m.

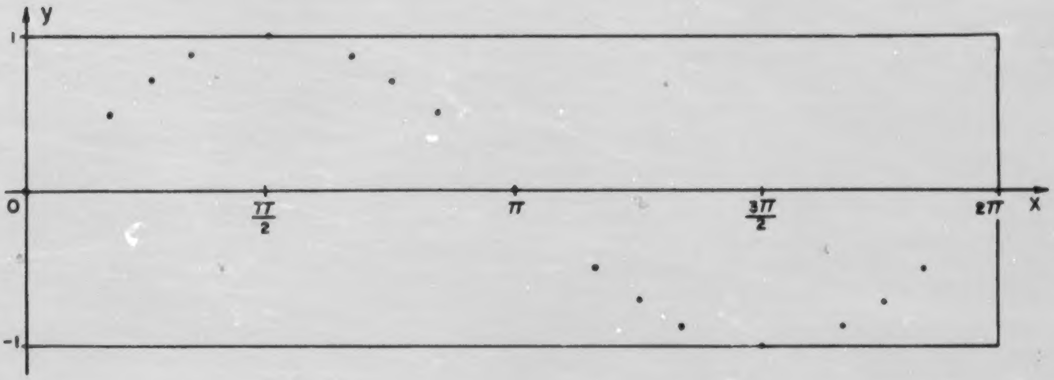

Figure 5-2k. Values of sin x: $x \rightarrow \sin x$

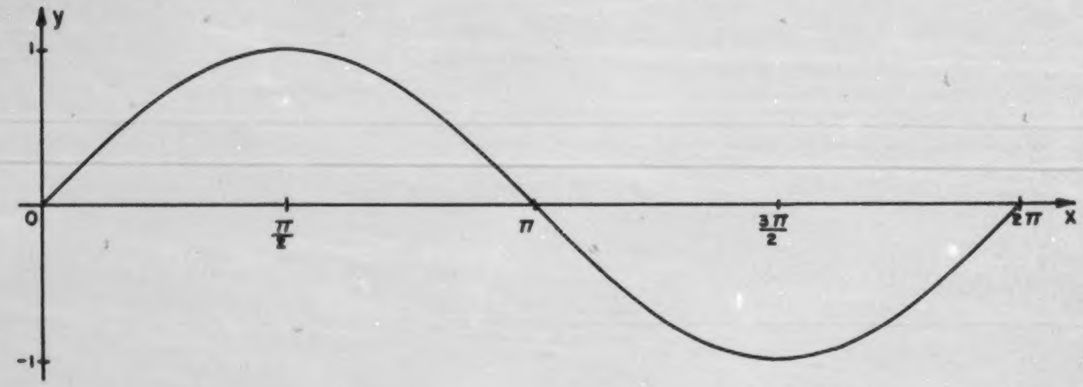

Figure 5-2l. Graph of one cycle of sin.

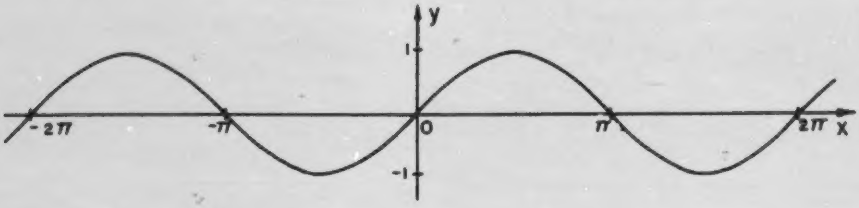

Figure 5-2m. Graph of sin.

Since it is often necessary to work with

$$y = A \cos x$$

$$y = \cos Bx \qquad (4)$$

$$y = \cos (x + C) \quad \text{(A, B, and C constants)}$$

or some combination of these expressions, it is worthwhile to inquire into the effect that these constants have on the behavior of y. In case of

$$y = A \cos x \qquad (A > 0),$$

the A simply multiplies each ordinate of $y = \cos x$ by A, and the graph of $y = A \cos x$ would appear as in Figure 5-2n.

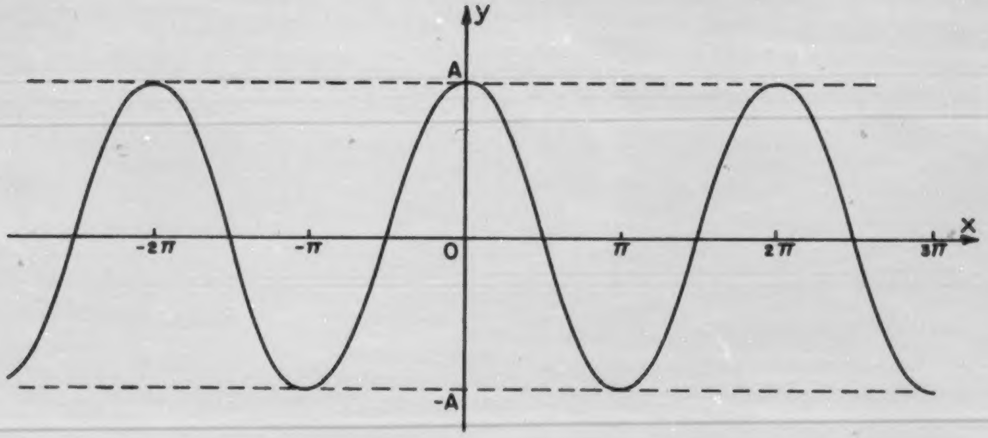

Figure 5-2n. Graph of $y = A \cos x$.

In Exercises 5, 6, and 7 you are asked to determine for yourself the effects of B and C in Equations (4).

Exercises 5-2

1. Using $f(x + 2n\pi) = f(x)$, and $f: x \rightarrow \cos x$, find

   a) $f(3\pi)$,
   b) $f(\frac{7\pi}{3})$,
   c) $f(\frac{9\pi}{2})$,
   d) $f(\frac{25\pi}{6})$,
   e) $f(-7\pi)$,
   f) $f(-\frac{10\pi}{3})$.

2. If $f: x \rightarrow \sin x$, find the values of f in Exercise 1 above.
3. For what values of x (if any) will
   a) $\sin x = \cos x$?
   b) $\sin x = -\cos x$?
   c) $\sin x = \sin (-x)$?
   d) $\cos x = \cos (-x)$?

4. Graph on the same set of axes the functions $f: x \rightarrow y$ defined by the following, using Table 5-1 to find values for the functions.
   a) $y = 2 \cos x$,
   b) $y = 3 \cos x$,
   c) $y = \frac{1}{2} \cos x$.
5. Repeat Exercise 4 using
   a) $y = \cos 2x$,
   b) $y = \cos 3x$,
   c) $y = \cos \frac{1}{2}x$.
6. Repeat Exercise 4 using
   a) $y = \cos (x + \frac{\pi}{2})$,
   b) $y = \cos (x - \frac{\pi}{2})$,
   c) $y = \cos (x + \pi)$.
7. From the results of Exercises 4, 5, and 6 above, what effect do you think the constant k will have on the graph of
   a) $y = k \cos x$?
   b) $y = \cos kx$?
   c) $y = \cos (x + k)$?
8. From the results of Exercise 6(b) above and Figure 5-2m, what can you say about $\cos (x - \frac{\pi}{2})$ and $\sin x$?
9. As explained in the text, symmetric points with respect to the u-axis on the unit circle $u^2 + v^2 = 1$ are obtained by proceeding a distance x in the clockwise and counterclockwise senses along the circle. In other words, if $\wp(x) = (u, v)$ then $\wp(-x) = (u, -v)$. It follows that

$$\cos x = \cos (-x)$$

$$\sin x = -\sin (-x)$$

   What relations between the circular functions can you derive in similar fashion from the following symmetries of the circle?
   a) The symmetry with respect to the origin.
   b) The symmetry with respect to the v-axis.

5-3. Angle and Angle Measure

As we remarked in Section 5-1, the circular functions are closely related to the functions of angles studied in elementary trigonometry. In a sense, all that we have done is to measure angles in a new way. To see precisely what the difference is, let us recall a few fundamentals.

An angle is defined in geometry as a pair of rays or half-lines with a common end point. (Figure 5-3a.) Let R_1 and R_2 be two rays originating at the point O. Draw any circle with O as center; denote its radius by r. The rays R_1 and R_2 meet the circle in two points P_1 and P_2 which divide the circle into two parts. Here we consider directed angles and distinguish between the angles defined by the pair R_1, R_2 according to their order. Specifically, we set $\alpha = \measuredangle (R_1, R_2)$ and $\beta = \measuredangle (R_2, R_1)$ where each angle includes that arc of the circle which is obtained by passing counterclockwise along the circle from the first ray of the pair to the second (Figure 5-3a).

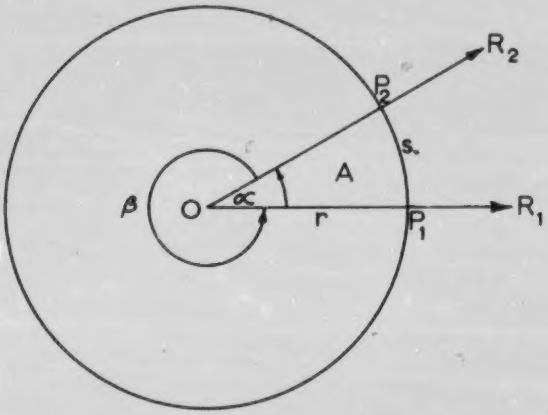

Figure 5-3a. Angles α and β.

In establishing degree measure, we could divide a circle into 360 equal units and measure an angle α by the number of units of arc it includes. For instance, if we found that an angle included $\frac{1}{3}$ of the circumference we would say that the angle measured $\frac{1}{3} \times 360^\circ$ or 120°. In general, if we divide the circumference of a circle into k equal parts, each of length $\frac{2\pi r}{k}$, then this length could be our unit of angle measure. Since the numerical factor $\frac{2\pi}{k}$ appears in many important formulas, it is useful to choose k so that the factor is 1. In order to do this, it is clear that k must equal 2π. In this case, $\frac{2\pi r}{k}$ will be equal to r, the radius of the circle. When $k = 2\pi$ we call the resulting unit of angle measure a radian. Radian measure is related to degree measure by

$$1 \text{ radian} = \left(\frac{360}{2\pi}\right)^\circ = \left(\frac{180}{\pi}\right)^\circ \qquad (1)$$

and

$$1^\circ = \frac{\pi}{180} \text{ radians.} \qquad (2)$$

You should note that this definition of the radian measure of an angle implies that an angle of 1 radian intercepts an arc of length s equal to r, the radius of the circle. In general an angle of x radians intercepts an arc of length xr. That is, $s = xr$ where x is the measure of the central angle in radians while s and r are the lengths of the arc and the radius measured in the same linear units.

In working with radian measure, it is customary simply to give the measure of an angle α as, say, $\frac{\pi}{2}$, rather than $\frac{\pi}{2}$ radians. If we use degree measure, however, the degree symbol will always be written, as for example, 90°, 45°, etc.

It is also possible to measure an angle α by the area A of the sector it includes (Figure 5-3a). Specifically, we have that the area A is the same fraction of the area of the interior of the circle as the arc s is of the circumference, that is,

$$\frac{A}{\pi r^2} = \frac{s}{2\pi r}. \qquad (3)$$

We saw above that the arc length s on a circle included by an angle α may be expressed as $s = rx$ where r is the radius of the circle and x is the radian measure of α. It follows from

(3) that

$$\frac{A}{\pi r^2} = \frac{x}{2\pi}$$

or

$$x = \frac{2A}{r^2} . \qquad (4)$$

That is, the measure x of α in radians is twice the area of the included sector divided by the square of the radius.

Exercises 5-3

1. Change the following radian measure to degree measure.
    - a) $\frac{2\pi}{3}$,
    - b) $\frac{\pi}{6}$,
    - c) $\frac{-2\pi}{3}$,
    - d) $\frac{7\pi}{6}$,
    - e) 2π,
    - f) $\frac{5\pi}{6}$,
    - g) $\frac{8\pi}{3}$,
    - h) $\frac{18\pi}{5}$,
    - i) $\frac{13\pi}{4}$.
2. Change the following degree measure to radian measure.
    - a) 270°,
    - b) -30°,
    - c) 135°,
    - d) 480°,
    - e) 195°,
    - f) -105°,
    - g) 810°,
    - h) 190°,
    - i) 18°.
3. What is the measure (in radians) of an angle which forms a sector of area 9π if the radius of the circle is 3 units?
4. What is the area of the sector formed by an angle of $(\frac{3}{2})\pi$, if the radius of the circle is 2 units?
5. Suppose that we wish to find a unit of measure so that a quarter of a circle will contain 100 such units.
    - a) How many such units will be equivalent to 1°?
    - b) How many such units will be equivalent to 1 radian?
    - c) How many of these units will a central angle contain, if the included arc is equal in length to the diameter of the circle?

5-4. Uniform Circular Motion

Let us again consider the motion of a point P around a circle of radius r in the uv-plane, and suppose that P moves at the constant speed of s units per second. Let $P_0(r, 0)$ represent the initial position of P. After one second, P will be at P_1, an arc-distance s away from P_0. After two seconds, P will be at P_2, an arc-distance $2s$ from P_0, and similarly after t seconds P will be at arc-distance ts. (Figure 5-4a.) Clearly $\measuredangle P_0OP_1 = \measuredangle P_1OP_2 = \measuredangle P_2OP_3 \ldots$

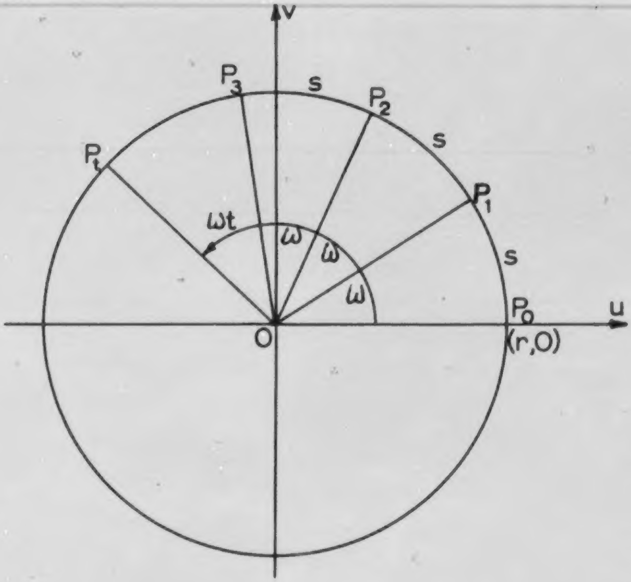

Figure 5-4a. Uniform motion of P on circle O.

. . and likewise for each additional second, since these central angles have equal arcs, each of length s. Each of these central angles may be written as $\omega = \frac{s}{r}$. After 2 seconds, OP will have rotated through an angle 2ω into position OP_2; after 3 seconds through an angle 3ω; and, in general, after t seconds through an angle of $t\omega$ or ωt. In other words, after t seconds, P

will have moved from $(r, 0)$ an arc-distance st, and OP_0 will have rotated from its initial position through an angle of ωt into the position OP. If we designate the coordinates of P by (u, v) we have

$$\begin{aligned} u &= r \cos \omega t \\ v &= r \sin \omega t. \end{aligned} \tag{1}$$

When $\omega t = 2\pi$, P will again be in the position P_0. This motion of the point from P_0 back into P_0 again is called a cycle. The time interval during which a cycle occurs is called the period; in this case, the period is $\frac{2\pi}{\omega}$. The number of cycles which occur during a fixed unit of time is called the frequency. Since we refer to the alternating current in our homes as "60-cycle", an abbreviation for "60 cycles per second", this notion of frequency is not altogether new to us.

To visualize the behavior of the point P in a different way, consider the motion of the point Q which is the projection of P on the v-axis. As P moves around the unit circle, Q moves up and down along a fixed diameter of the circle, and a pencil attached to Q will trace this diameter repeatedly -- assuming that the paper is fixed in position. If, however, the strip of paper is drawn from right to left at a constant speed, then the pencil will trace a curve, something like Figure 5-4b.

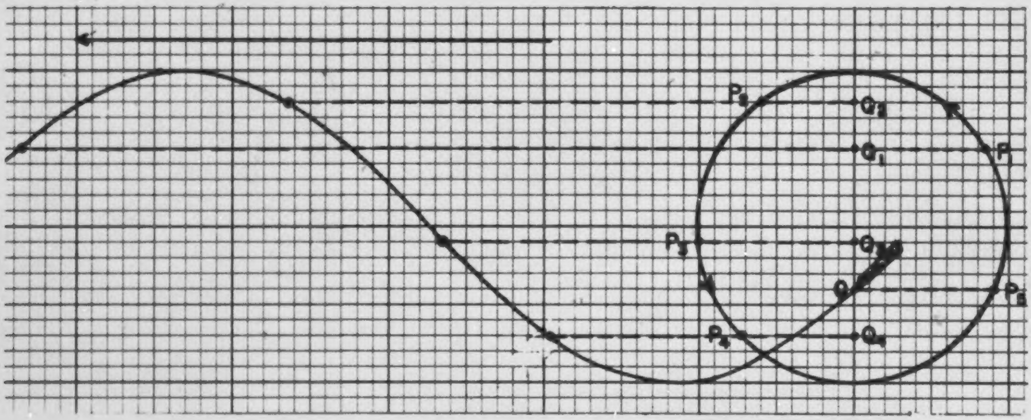

Figure 5-4b. Wave Motion

An examination of this figure will show why motion of this type is called wave motion. We note that the displacement y of Q from its central position is functionally related to the time t, that is, there is a function f such that $y = f(t)$. By suitably locating the origin of the ty-plane, we may have either $y = \cos \omega t$ or $y = \sin \omega t$; thus either of these equations may be looked upon as describing a pure wave or, as it is sometimes called, a simple harmonic motion. The surface of a body of water displays a wave motion when it is disturbed. Another familiar example is furnished by the electromagnetic waves used in radio, television, and radar, and modern physics has even detected wave-like behavior of the electrons of the atom.

One of the most interesting applications of the circular functions is to the theory of sound (acoustics). A sound wave is produced by a rapid alternation of pressure in some medium. A pure musical tone is produced by any pressure wave which can be described by a circular function of time, say:

$$p = A \sin \omega t \tag{2}$$

where p is the pressure at time t and the constants A and ω are positive. The equation (2) for the acoustical pressure, p, is exactly in the form of one of the equations of (1) even though no circular motion is involved; all that occurs is a fluctuation of the pressure at a given point of space.* Here the numbers A and ω have direct musical significance. The number A is called the amplitude of the wave; it is the peak pressure and its square is a measure of the loudness. The number ω is proportional to the frequency and is a measure of pitch; the larger ω the more shrill the tone.

The effectiveness of the application of circular functions to the theory of sound stems from the principle of superposition. If two instruments individually produce acoustical pressures p_1 and p_2 then together they produce the pressure $p_1 + p_2$. If

---

*The acoustical pressure is defined as the difference between the gas pressure in the wave and the pressure of the gas if it is left undisturbed.

p_1 and p_2 have a common period then the sum $p_1 + p_2$ has the same period. This is the root of the principle of harmony; if two instruments are tuned to the same note, they will produce no strange new note when played together.

Let us suppose, for example, that two pure tones are produced with individual pressure waves of the same frequency, say

$$u = A \cos \omega t \tag{3}$$

$$v = B \sin \omega t \tag{4}$$

where A, B and ω are positive. According to the principle of superposition, the net pressure is

$$p = A \cos \omega t + B \sin \omega t.$$

What does the graph of this equation look like? We shall answer this question by reducing the problem to two simpler problems, that is, of graphing (3) and (4) above. For each t, the value of p is obtained from the individual graphs, since

$$p = u + v.$$

To illustrate these ideas with specific numerical values in place of A, B and ω; let

$$A = 3, \quad B = 4, \quad \omega = \pi.$$

Then we wish to graph

$$p = 3 \cos \pi t + 4 \sin \pi t. \tag{5}$$

Equations (3) and (4) become

$$u = 3 \cos \pi t, \tag{6}$$

$$v = 4 \sin \pi t. \tag{7}$$

By drawing the graphs of (6) (Figure 5-4c) and (7) (Figure 5-4d) on the same set of axes, and by adding the corresponding ordinates of these graphs at each value of t, we obtain the graph of (5) shown in Figure 5-4e. You will notice that certain points on the graph of p are labeled with their coordinates. These are points which are either easy to find, or which have some special interest.

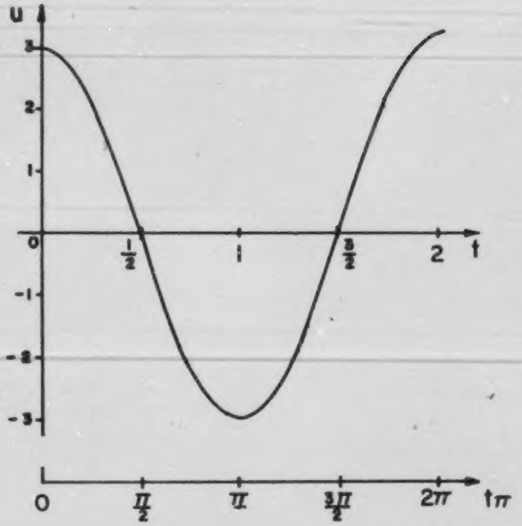

Figure 5-4c. Graph of
$u = 3 \cos \pi t$.

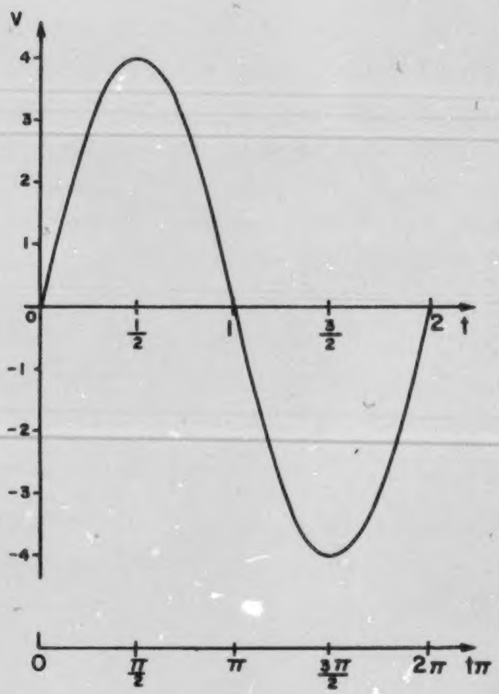

Figure 5-4d. Graph of
$v = 4 \sin \pi t$.

The points (0, 3), (0.5, 4), (1, -3), (1.5, -4) and (2, 3) are easy to find since they are the points where either $u = 0$ or $v = 0$. The points (0.29, 5) and (1.29, -5) are important because they represent the first maximum and minimum points on the graph of p, while (0.79, 0) and (1.79, 0) are the first zeros of p. To find the maximum and minimum points and zeros of p involves the use of tables and hence we shall put off a discussion of this matter until Section 5-7, although a careful graphing should produce fairly good approximations to them.

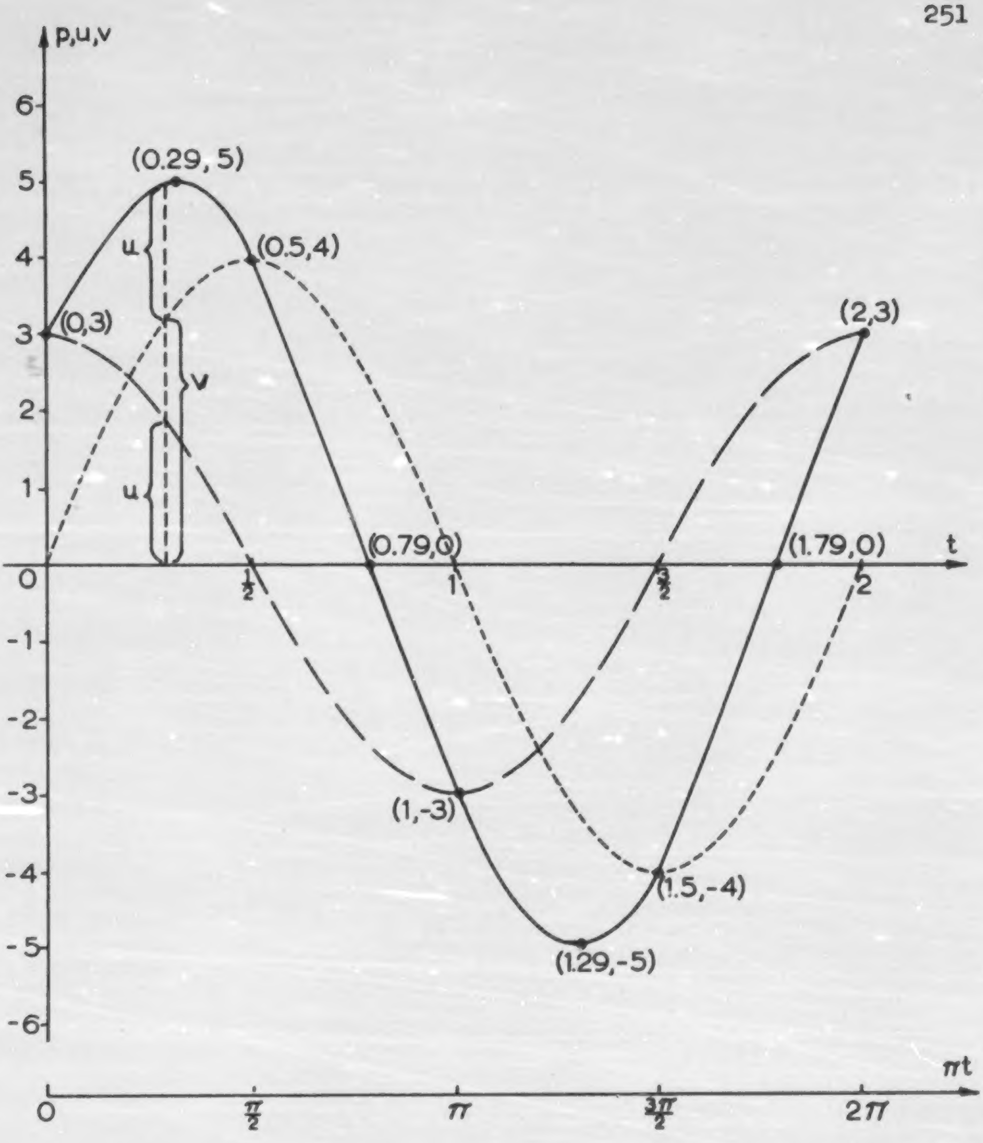

Figure 5-4e. The sum of two pure waves of equal period.

Dashed curve: $u = 3 \cos \pi t$. Dotted curve: $v = 4 \sin \pi t$. Full curve: $p = 3 \cos \pi t + 4 \sin \pi t$; $0 \leq t \leq 2$. (The scales are not the same on the two axes; this distortion is introduced in order to show the details more clearly.)

Exercises 5-4

1. Extend the three curves in Figure 5-4e to the interval $|t| \leq 2$. To the interval $|t| \leq 3$. What do you observe about the graph of $p = 3 \cos \pi t + 4 \sin \pi t$ over $|t| \leq 3$? Is it periodic? What is its period? Give reasons for your answers.
2. Sketch graphs of each of the following curves over one complete cycle; and state what the period is, and what the range is, if you can.
   a) $y = 2 \sin 3t$.
   b) $y = -3 \sin 2t$.
   c) $y = 4 \cos (\frac{x}{2})$.
   d) $y = 3 \cos (-x)$.
   e) $y = 2 \sin x - \cos x$.

5-5. Vectors and Rotations

In the next section, we shall develop the important formulas for $\sin(x + y)$ and $\cos(x + y)$. Because our development will rely on certain properties of plane vectors, we give, in this section, an informal summary of those properties.

You have probably encountered vectors in your earlier work in mathematics or science. The physicist uses them to represent quantities such as displacements, forces, and velocities, which have both magnitude and direction. Some examples of vector quantities are the velocity of a train along a track or of the wind at a given point, the weight of a body (the force of gravity), and the displacement from the origin of a point in the Cartesian plane.

In a two-dimensional system, it is often convenient to represent vectors by arrows (which have both a length, representing magnitude, and a direction) and to use geometrical language. We shall do this, and we shall restrict ourselves to vectors all of which start from a single point; in our discussion we shall take this point to be the origin. If S and T are vectors, we define the sum $S + T$ to be the vector R represented by the diagonal of the parallelogram which has sides S and T, as shown in Figure 5-5a. If T is a vector and a is a number, then we define the product aT to be a vector whose magnitude is |a| times that of T and whose direction is the same as T if a > 0 and opposite to T if a < 0; in either case, T and aT are collinear. Figure 5-5b illustrates this for a = 2 and a = -2. It is an experimental fact that these definitions correspond to physical reality; the net effect of two forces acting at a point, for example, is that of a single force determined by the parallelogram law of addition.

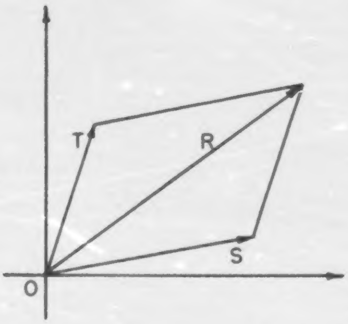

Figure 5-5a. The sum of two vectors.

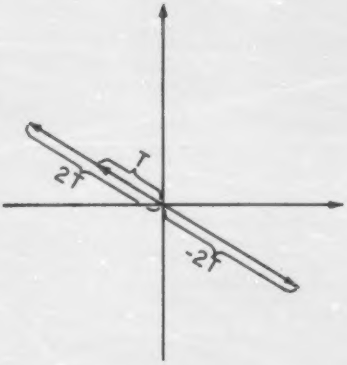

Figure 5-5b. A vector multiplied by a number.

These definitions of vector sum and of multiplication by a number make it possible to express all plane vectors from the origin in terms of two basic vectors. It is convenient to take as these basic vectors the vector U from the origin to (1, 0) and the vector V from the origin to (0, 1). Then, for any vector R, there exist unique numbers u and v such that

$$R = uU + vV; \qquad (1)$$

in fact, the numbers u and v are precisely the coordinates of the tip of the arrow representing R (Figure 5-5c). To take a specific example, the vector S from the origin to the point $P(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ can be expressed in terms of the basic vectors U and V as

$$S = -\frac{1}{2}U + \frac{\sqrt{3}}{2}V,$$

as shown in Figure 5-5d.

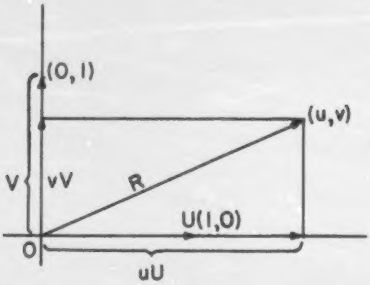

Figure 5-5c. A vector in terms of the basic vectors U and V.

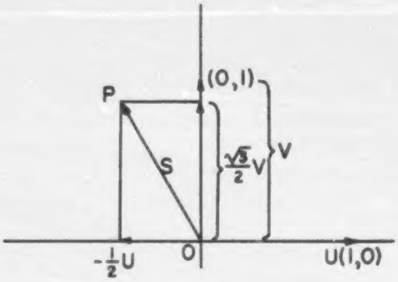

Figure 5-5d. $S = -\frac{1}{2}U + \frac{\sqrt{3}}{2}V$.

We now introduce the idea of a rotation of the whole plane about the origin O. Such a rotation carries each vector into a unique vector, and we may therefore regard it as a function whose domain and range are sets of vectors. We have so far in this course considered mostly functions which map numbers onto numbers, but it will be useful, in this section, to think of a rotation as a new kind of function which maps vectors into vectors.

Any rotation of the sort we are considering is completely specified by the length x of the arc AP of the unit circle through which the rotation carries the point A(1, 0). Let f be the rotation (function) which maps the vector OA (that is, U) onto the vector OP whose tip P has coordinates (u, v). As we have seen above, OP can be expressed in terms of the basic vectors U and V as uU + vV. Hence

$$f(U) = OP = uU + vV, \tag{2}$$

as pictured in Figure 5-5e. The same rotation f carries the point $B(0, 1)$ into the point $Q(-v, u)$, as can be shown by congruent triangles (see Figure 5-5e), so we also have

$$f(V) = OQ = -vU + uV. \tag{3}$$

(The figure is valid only when $0 < x < \frac{\pi}{2}$. The result, however, is true for any real x; for a more general derivation, see Exercises 8 and 9.)

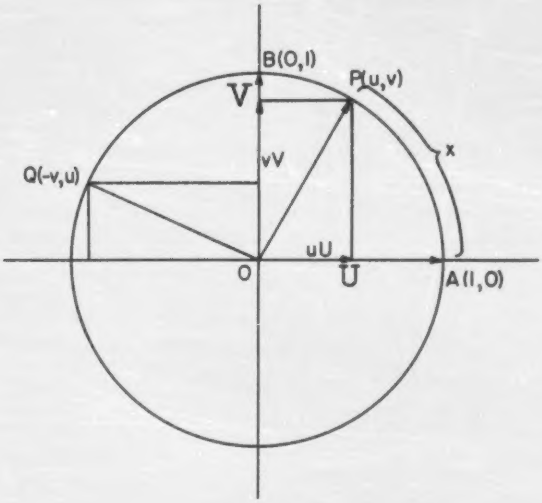

Figure 5-5e. The effect of a rotation on the basic vectors U and V.

Now suppose that we subject the plane to a second rotation g, in which points on the unit circle are displaced through an arc of length y. Since g also is a function, we may regard the successive applications of the rotations f and g as a composite function gf, as in Section 1-5. From Equation (2) and the definition of composition, we have

$$(gf)(U) = g(f(U)) = g(OP) = g(uU + vV). \tag{4}$$

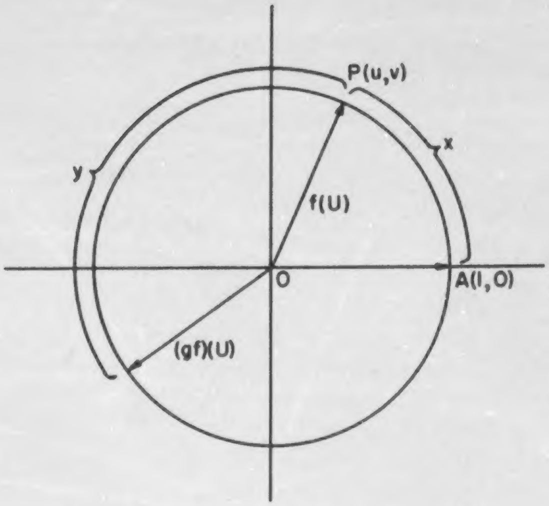

Figure 5-5f.

We must now pay some attention to two important properties of rotations. First, a rotation does not change the angle between two vectors, and collinear vectors will therefore be rotated into collinear vectors. Second, a rotation does not change the length of any vector. Now, if a is a number $(\neq 0)$ and T is a vector, then the vector aT is collinear with T. If f is a rotation, the two stated properties ensure that T and $f(T)$ have the same length, that aT and $f(aT)$ have the same length, and that $f(aT)$ is collinear with $f(T)$. We will therefore get the same vector from T if we first multiply by a and then rotate, or first rotate and then multiply by a:

$$f(aT) = af(T). \tag{5}$$

The same two properties of rotations also ensure that a parallelogram will not be distorted by a rotation. Since the addition of vectors is defined in terms of parallelograms, it follows that rotations preserve sums; that is, if f is a rotation, and if S and T are vectors, then

$$f(S + T) = f(S) + f(T). \tag{6}$$

From (5) and (6),

$$g(uU + vV) = ug(U) + vg(V),$$

and we may therefore rewrite (4) as

$$(gf)(U) = ug(U) + vg(V). \qquad (7)$$

Exercises 5-5

1. Let T be the vector OP where P is the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Write T in the form $uU + vV$. If $T = f(U)$, find the arc on the unit circle which specifies the rotation f.
2. In Exercise 1, replace P by
   (a) the point $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$,
   (b) the point $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$.
3. Find $f(U)$ if the rotation f is specified by an arc of the unit circle which is
   (a) $\frac{3\pi}{2}$ units long.
   (b) 2π units long.
4. Write $f(U)$ in the form $uU + vV$ if f corresponds to an arc of the unit circle which is
   (a) $\frac{\pi}{4}$ units long.
   (b) $\frac{\pi}{3}$ units long.
5. Do Exercise 4 for an arc $\frac{3\pi}{4}$ units long.
6. Let f correspond to a rotation of $\frac{\pi}{2}$ units and g to a rotation of $\frac{\pi}{4}$ units. Show that, since $V = f(U)$, the result in Exercise 5 is equivalent to $g(V)$.
7. If f and g are any two rotations of the plane about the origin, show that $fg = gf$.
8. If the rotation f corresponds to an arc x and the rotation g to an arc $\frac{\pi}{2}$, show that $f(V) = (fg)(U) = (gf)(U)$.
9. In Exercise 8, put $f(U) = uU + vV$, and hence show that $f(V) = g(uU) + g(vV) = ug(U) + vg(V) = uV - vU$.

5-6. The Addition Formulas

We are now ready to bring the circular functions into the picture. Since f maps the vector OA onto OP so that $A(1, 0)$ is carried through an arc x of the unit circle to $P(u, v)$, it follows from the definitions of Section 5-1 that

$$u = \cos x \qquad \text{and} \qquad v = \sin x.$$

Hence Equations (2) and (3) of Section 5-5 can be written

$$f(U) = (\cos x)U + (\sin x)V \tag{1}$$

and

$$f(V) = (-\sin x)U + (\cos x)V\,. \tag{2}$$

Since, moreover, the rotation g differs from the rotation f only in that the arc length involved is y instead of x, we may similarly write

$$g(U) = (\cos y)U + (\sin y)V \tag{3}$$

and

$$g(V) = (-\sin y)U + (\cos y)V. \tag{4}$$

Substituting these results in (7) of Section 5-5 gives us

$$\begin{aligned}(gf)(U) &= (\cos x)\big((\cos y)U + (\sin y)V\big) \\ &\qquad + (\sin x)\big((-\sin y)U + (\cos y)V\big) \\ &= (\cos x \cos y - \sin x \sin y)U \\ &\qquad + (\sin x \cos y + \cos x \sin y)V. \end{aligned} \tag{5}$$

Furthermore, the composite rotation gf can be regarded as a single rotation through an arc of length $x + y$, and we may therefore write, by analogy with (1),

$$(gf)(U) = \big(\cos(x + y)\big)\, U + \big(\sin(x + y)\big)\, V. \tag{6}$$

We now have, in (5) and (6), two ways of expressing the vector $(gf)(U)$ in terms of the basic vectors U and V. Since there is essentially only one such way of expressing any vector, it follows that the coefficient of U in (5) must be the same as the coefficient of U in (6), or

$$\cos(x + y) = \cos x \cos y - \sin x \sin y, \tag{7}$$

and a similar comparison of the coefficients of V in the two expressions yields

$$\sin(x + y) = \sin x \cos y + \cos x \sin y. \qquad (8)$$

These are the desired addition formulas for the sine and cosine.

We also obtain the subtraction formulas very quickly from Equations (7) and (8). Thus

$$\cos(x - y) = \cos\big(x + (-y)\big) = \cos x \cos(-y) - \sin x \sin(-y). \qquad (9)$$

Since, however, (Section 5-2, Equations (3))

$$\cos(-y) = \cos y$$

$$\text{and} \qquad \sin(-y) = -\sin y,$$

we may write (9) as

$$\cos(x - y) = \cos x \cos y + \sin x \sin y. \qquad (10)$$

In the Exercises, you will be asked to show similarly that

$$\sin(x - y) = \sin x \cos y - \cos x \sin y. \qquad (11)$$

From Formulas (7) and (8) and (10) and (11), it is easy to derive a large number of familiar trigonometric formulas.

Example. Find $\cos(x + \pi)$ and $\sin(x + \pi)$.
Solution. By (7), with $y = \pi$,

$$\cos(x + \pi) = \cos x \cos \pi - \sin x \sin \pi.$$

Now, $\cos \pi = -1$ and $\sin \pi = 0$. Hence $\cos(x + \pi) = -\cos x$.
Similarly, from (8),

$$\begin{aligned}\sin(x + \pi) &= \sin x \cos \pi + \cos x \sin \pi \\ &= \sin x(-1) + \cos x(0) \\ &= -\sin x.\end{aligned}$$

Exercises 5-6

1. By use of the appropriate sum or difference formula show that
   a) $\cos(\frac{\pi}{2} - x) = \sin x$,
   b) $\sin(\frac{\pi}{2} - x) = \cos x$,
   c) $\cos(x + \frac{\pi}{2}) = -\sin x$,

d) $\sin(x + \frac{\pi}{2}) = \cos x$,

e) $\cos(\pi - x) = -\cos x$,

f) $\sin(\pi - x) = \sin x$,

g) $\cos(\frac{3\pi}{2} + x) = \sin x$,

h) $\sin(\frac{3\pi}{2} + x) = -\cos x$,

i) $\sin(\frac{\pi}{4} + x) = \cos(\frac{\pi}{4} - x)$.

2. Prove that $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

*3. Show that formulas (7), (8), and (11) may all be obtained from formula (10), and, hence, that all of the relationships mentioned in this section follow from formula (10).

4. Prove that the function tangent (abbreviated tan) defined by

$$\tan: x \longrightarrow \frac{\sin x}{\cos x} \quad (x \neq \pm \frac{\pi}{2} + 2n\pi)$$

is periodic, with period π. Why are the values $\pm \frac{\pi}{2} + 2n\pi$ excluded from the domain of the tangent function?

5. Using the definition of the function tangent in Exercise 4 and the formulas (7), (8), (10), (11), develop formulas for $\tan(x + y)$ and $\tan(x - y)$ in terms of $\tan x$ and $\tan y$.

6. Using the results of Exercise 5, develop formulas for $\tan(\pi - x)$ and $\tan(\pi + x)$. Also show that $\tan(-x) = -\tan x$.

7. Express $\sin 2x$, $\cos 2x$ and $\tan 2x$ in terms of functions of x. (Hint: Let $y = x$ in the appropriate formulas.)

8. Express $\sin 3x$ in terms of functions of x.

9. In Exercise 7 you were asked to express $\cos 2x$ in terms of functions of x. One possible result is $\cos 2x = 1 - 2 \sin^2 x$. In this expression substitute $x = \frac{y}{2}$ and solve for $\sin \frac{y}{2}$.

10. In Exercise 9, $\cos 2x$ may also be written as $2 \cos^2 x - 1$. Use this formula to get a formula for $\cos \frac{y}{2}$.

11. Using the definitions of the function tan and the results of Exercises 9 and 10, derive a formula for $\tan \frac{y}{2}$. This will be an expression involving radicals, but by rationalizing

in succession the numerator and the denominator you can get two different expressions for $\tan \frac{y}{2}$, not involving radicals.

In Section 5-5 we developed the algebra of rotations, and in this section we have applied this algebra to derive the addition formulas for the sine and cosine functions. As we shall now indicate, there is a close parallel between the algebra of rotations and the algebra of complex numbers.

If two complex numbers are expressed in polar form, as are

$$z_1 = r_1(\cos x_1 + i \sin x_1)$$

and

$$z_2 = r_2(\cos x_2 + i \sin x_2)$$

then their product can be found by multiplying their absolute values r_1 and r_2, and adding their arguments, x_1 and x_2:

$$z_1 z_2 = r_1 r_2 \big(\cos(x_1 + x_2) + i \sin(x_1 + x_2)\big).$$

Multiplying any complex number z by the special complex number

$$\cos x + i \sin x = 1(\cos x + i \sin x)$$

is therefore equivalent to leaving the absolute value of z unchanged and adding x to the argument of z. Hence, if we represent z by a vector in the complex plane, then multiplying by $\cos x + i \sin x$ is equivalent to rotating this vector through an arc x, as in Section 5-5.

Let us replace the vector U of Section 5-5 by the complex number

$$1 = \cos 0 + i \sin 0.$$

Then the product

$$(\cos x + i \sin x)\cdot 1 = \cos x + i \sin x$$

represents the vector formerly called $f(U)$ (see Figure 5-6a), and $(gf)(U)$ becomes

$$(\cos y + i \sin y)(\cos x + i \sin x)1$$
$$= \big((\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y)\big)\cdot 1.$$

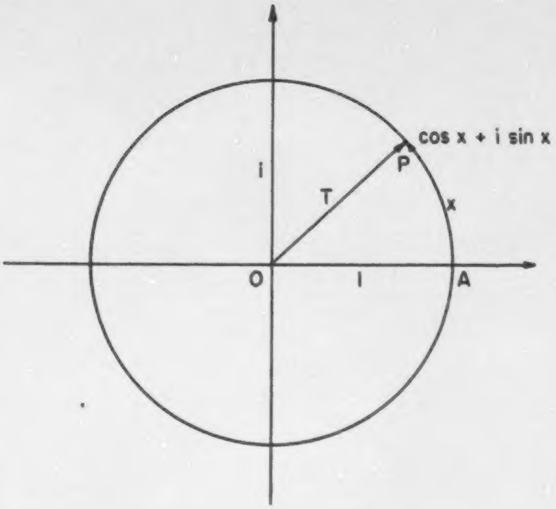

Figure 5-6a. Representation of $T = \cos x + i \sin x$.

If we replace $(gf)(U)$ by

$$\bigl(\cos(x + y) + i \sin(x + y)\bigr) \cdot 1$$

we have

$$\cos(x + y) + i \sin(x + y) = (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y).$$

By equating real and imaginary parts we obtain the addition formulas (7) and (8).

The subtraction formulas may be derived equally simply. Since g^{-1} is equivalent to rotating through an angle $-y$, we have $(g^{-1}f)(U) = g^{-1}\bigl(f(U)\bigr)$ and therefore

$$\bigl(\cos(x - y) + i \sin(x - y)\bigr) \cdot 1.$$

Hence

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

and

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

5-7. Construction and Use of Tables of Circular Functions

It would be difficult to give in a short span an indication of the enormous variety of ways in which the addition formulas of Section 5-6,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (1)$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad (2)$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (3)$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad (4)$$

turn up in mathematics and in the application of mathematics to the sciences. In this section and in Sections 5-8 and 5-9, we shall describe some of the more common applications. The first of these is a table of values of the sine and cosine functions.

In Exercise 1 of Section 5-6, you used the difference formulas to show that

$$\sin(\tfrac{\pi}{2} - x) = \cos x$$

and

$$\cos(\tfrac{\pi}{2} - x) = \sin x.$$

These formulas permit the tabulation of $\sin x$ and $\cos x$ in a very neat way. If we had a table of cosines for $0 \leq x \leq \frac{\pi}{2}$, this would, in effect, give a table of sines in backward order. For example, from the table of special values in Section 5-2, we obtain the sample table shown, where $y = \frac{\pi}{2} - x$.

x	$\cos x$	$\frac{\pi}{2} - x$
0	1	$\frac{\pi}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\pi}{6}$
$\frac{\pi}{2}$	0	0
$\frac{\pi}{2} - y$	$\sin y$	y

In this table the values of the cosine are read from the top down and the values of the sine from the bottom up. Since it is a very inefficient use of space to put so few columns on a page, the table is usually folded in the middle about the value $x = y = \frac{\pi}{4}$ and is constructed as in the following sample:

x	$\cos x$	$\sin x$	$\frac{\pi}{2} - x$
0	1	0	$\frac{\pi}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\pi}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
$\frac{\pi}{2} - y$	$\sin y$	$\cos y$	y

At the end of the chapter we give three tables:

I. A table of $\sin x$ and $\cos x$ for decimal values of x up to 1.57 (slightly less than $\frac{\pi}{2}$).

II. A table of $\sin \frac{\pi x}{2}$ and $\cos \frac{\pi x}{2}$ in decimal fractions of $\frac{\pi}{2}$ up to 1.00.

III. A table of $\sin^{\circ} x$, $\cos^{\circ} x$ and $\tan^{\circ} x$, in degrees up to 90°.

(We define $\sin^{\circ}: x^{\circ} \longrightarrow \sin x$, with similar definitions for $\cos^{\circ}$ and $\tan^{\circ}$. It is usual to write $\sin x$ in place of $\sin^{\circ} x$, etc., when the context makes it clear what is intended. We shall follow this practice.)

Exercises 5-7a

1. Why is Table I not folded as are Tables II and III?
2. Find from Table I, $\sin x$ and $\cos x$ when x is equal to
   a) 0.73 c) 1.55
   b) -5.17 d) 6.97 (Hint: $2\pi \approx 6.28$)
3. From Table I, find x when $0 \leq x \leq \frac{\pi}{2}$ and
   a) $\sin x \approx 0.1099$ c) $\sin x \approx 0.6495$
   b) $\cos x \approx 0.9131$ d) $\cos x \approx 0.5403$

4. From Table II, find $\sin \omega t$ and $\cos \omega t$ if $\omega = \frac{\pi}{2}$ and

a) $t = 0.31$

b) $t = 0.79$

c) $t = 0.62$

d) $t = 0.71$

5. From Table II, find t (interpolating, if necessary), if $\omega = \frac{\pi}{2}$, $0 \leq t \leq 1$ and

a) $\sin \omega t \approx 0.827$

b) $\cos \omega t \approx 0.905$

c) $\sin \omega t \approx 0.475$

d) $\cos \omega t \approx 0.795$

6. From Table III, find $\sin x$ and $\cos x$ (interpolating, if necessary) when

a) $x = 45^\circ$

b) $x = 73^\circ$

c) $x = 36.2^\circ$

d) $x = 81.5^\circ$

7. From Table III, find x when $0 \leq x \leq 90^\circ$ and

a) $\sin x \approx 0.629$

b) $\cos x \approx 0.991$

c) $\sin x \approx 0.621$

d) $\cos x \approx 0.895$

Extending the scope of the tables. Table I, at the end of this chapter, gives values of the circular functions $\cos: x \longrightarrow \cos x$ and $\sin: x \longrightarrow \sin x$ only for $0 \leq x < \frac{\pi}{2}$, but we can extend its scope to the set of all real numbers by using (a) Equations (1) - (4), (b) our knowledge of the circular functions of all multiples of $\frac{\pi}{2}$ (see, for example, Table 5-1), and (c) the fact that any real number can be expressed as the sum (or difference) of two numbers of which one is a multiple of $\frac{\pi}{2}$ and the other is in the interval $\{x: 0 \leq x < \frac{\pi}{2}\}$. Similar remarks apply to Tables II and III. The technique is best explained through examples.

Example 1. Find $\sin 2$.

Solution. Since $\frac{\pi}{2} \approx 1.57$, we write $2 = 1.57 + 0.43$, and, using Equation (2), we then have

$$\begin{aligned} \sin 2 &= \sin(1.57 + 0.43) \\ &\approx \sin(\tfrac{\pi}{2} + 0.43) = \sin \tfrac{\pi}{2} \cos 0.43 + \cos \tfrac{\pi}{2} \sin 0.43 \\ &= \cos 0.43 \approx 0.9090. \end{aligned}$$

Alternatively, $2 = 3.14 - 1.14 \approx \pi - 1.14$, and therefore

$$\sin 2 \approx \sin(\pi - 1.14) = \sin \pi \cos 1.14 - \cos \pi \sin 1.14$$
$$= \sin 1.14 \approx 0.9086.$$

Example 2. Find $\cos 4.56$.

Solution. Since $4.56 = 3.14 + 1.42 \approx \pi + 1.42$, we have

$$\cos 4.56 \approx \cos(\pi + 1.42) = \cos \pi \cos 1.42 - \sin \pi \sin 1.42$$
$$= -\cos 1.42 \approx -0.1502.$$

This technique can be used to simplify expressions of the form $\sin (n\frac{\pi}{2} \pm x)$ and $\cos(n\frac{\pi}{2} \pm x)$.

Example 3. Simplify $\cos(\frac{5\pi}{2} + x)$.

Solution. $\cos (\frac{5\pi}{2} + x) = \cos \frac{5\pi}{2} \cos x - \sin \frac{5\pi}{2} \sin x$

$$= \cos \frac{\pi}{2} \cos x - \sin \frac{\pi}{2} \sin x$$
$$= -\sin x.$$

Example 4. Find $\cos 0.82\pi$.

Solution. In this case, it is easier to use Table II. Since $0.82\pi = 0.50\pi + 0.32\pi$, we have

$$\cos 0.82\pi = \cos(\frac{\pi}{2} + 0.32\pi)$$
$$= \cos \frac{\pi}{2} \cos 0.32\pi - \sin \frac{\pi}{2} \sin 0.32\pi$$
$$= -\sin 0.32\pi = -\sin 0.64(\frac{\pi}{2}) \approx -0.844.$$

Exercises 5-7b

Using the table that you think most convenient, find

1. $\sin 1.73$
2. $\cos 1.3\pi$
3. $\sin(-.37)$
4. $\sin(-.37\pi)$
5. $\cos 2.8\pi$
6. $\cos 1.8\pi$
7. $\cos 3.71$
8. $\sin 135^\circ$
9. $\cos(-135^\circ)$
10. $\sin 327^\circ$
11. $\cos(-327^\circ)$
12. $\cos 12.4\pi$
13. $\sin 12.4$

*14. $\cos(\sin .3\pi)$

*15. $\sin(\sin .7)$

5-8. Pure Waves: Frequency, Amplitude and Phase

As we remarked in Section 5-4, the superposition of two pure waves of the same frequency yields a pure wave of the given frequency. Now we shall be able to prove this result. In order to be more specific, instead of assuming that either of Equations (1) in Section 5-4 defines a pure wave, let us say that, by definition, a pure wave will have the form

$$y = A\cos(\omega t - \alpha), \tag{1}$$

where A and ω are positive and $0 \le \alpha < 2\pi$. The number α is called the phase of the pure wave. The sine function now becomes simply a special case of (1), and defines a pure wave with phase $\frac{\pi}{2}$,

$$y = \sin\omega t = \cos(\omega t - \tfrac{\pi}{2}). \tag{2}$$

The phase of a pure wave has a simple interpretation. We will take the graph of

$$y = \cos\omega t \tag{3}$$

as a standard of reference, and the cycle over the interval $(0 \le t < \frac{2\pi}{\omega})$ between two peaks of (3) as the standard cycle. Now the graph of $y = A\cos(\omega t - \alpha)$ reaches its peak, corresponding to the first peak of its standard cycle, at the point where $\omega t - \alpha = 0$, that is, at $t = \frac{\alpha}{\omega}$. Since $\frac{\alpha}{\omega}$ is positive, it is clear that the wave (1) reaches its first peak after the standard wave (3) reaches its first peak, since (3) has a peak at $t = 0$. That is, the wave (1) lags behind the wave (3) by an amount $\frac{\alpha}{\omega}$. Since the period of (3) is $\frac{2\pi}{\omega}$, this lag amounts to the fraction

$$\frac{\frac{\alpha}{\omega}}{\frac{2\pi}{\omega}} = \frac{\alpha}{2\pi}$$

of a period. (Figure 5-8a.) We see from (2) that $\sin\omega t$ lags behind $\cos\omega t$ by a quarter-period. (See Figure 5-4c and d.)

We now wish to test the idea that the sum of two pure waves which have the same period but differ in amplitude and phase, is again a pure wave of the same period with some new amplitude and phase. You will recall that in Section 5-4 we sketched the graph of

$$y = 3\cos\pi t + 4\sin\pi t \tag{4}$$

by adding the ordinates of the graphs of $u = 3\cos\pi t$ and $v = 4\sin\pi t$. The graph supported this idea. At that time we also had to leave open the question of the exact location of the maximum and minimum points and the zeros of the graph.

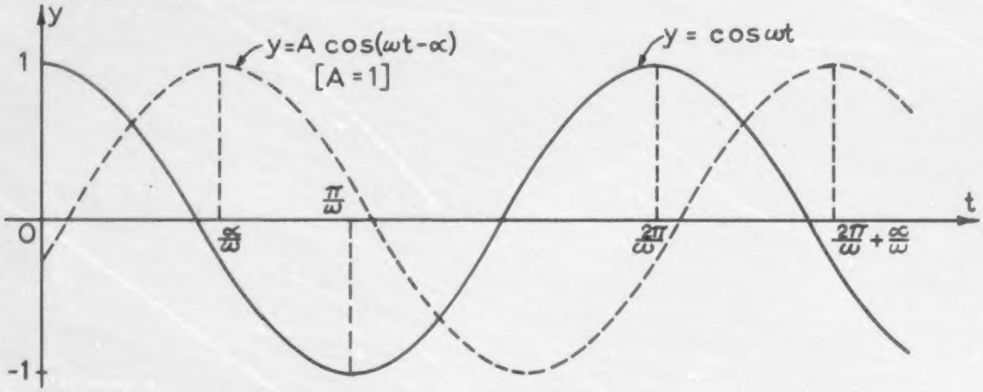

Figure 5-8a. Graphs of two cosine curves.

We are now in a position to deal with these problems. Since finding the maximum and minimum points and finding the zeros involve essentially the same procedure, we shall confine our attention to the maximum and minimum points.

Our basic problem still is to express

$$y = 3\cos\pi t + 4\sin\pi t$$

in the form of

$$y = A\cos(\omega t - \alpha) \tag{1}$$

that is, to show that y is a pure wave, but in the process we shall be able to obtain the exact location of the maximum and minimum points of the graph of the sum. If we write out (1) in terms of the formula

$$\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha \tag{5}$$

we obtain

$$y = A\cos(\omega t - \alpha) = A(\cos\omega t \cos\alpha + \sin\omega t \sin\alpha)$$

or

$$y = A\cos\omega t \cos\alpha + A\sin\omega t \sin\alpha. \qquad (6)$$

In our case, $\omega = \pi$ and we have

$$y = A\cos\pi t \cos\alpha + A\sin\pi t \sin\alpha. \qquad (7)$$

Upon comparing (7) with (4), we note that if

$$A\cos\alpha = 3 \quad \text{and} \quad A\sin\alpha = 4 \qquad (8)$$

then (7) and (4) will be identical. We shall therefore seek values of A and α which satisfy the Equations (8). To do this, we may begin by squaring the Equations (8) and adding them, to obtain

$$3^2 + 4^2 = A^2\cos^2\alpha + A^2\sin^2\alpha$$

$$9 + 16 = A^2(\cos^2\alpha + \sin^2\alpha)$$

or

$$A^2 = 25.$$

Since A is positive, we have

$$A = 5, \qquad (9)$$

and consequently from (8),

$$\cos\alpha = \frac{3}{5} \quad \text{and} \quad \sin\alpha = \frac{4}{5}. \qquad (10)$$

From Table I

$$\alpha \approx 0.927. \qquad (11)$$

Now, by using (9) and (11), we may put (4) in the form

$$y = 3\cos\pi t + 4\sin\pi t \approx 5\cos(\pi t - 0.927), \qquad (12)$$

showing that it is a pure wave with amplitude 5, period 2 (as before), and phase 0.927. We note that $\frac{0.927}{\pi} \approx 0.295$ is very close to the value 0.29 obtained graphically in Section 5-4. We are also in a position to locate the maximum and minimum points of our graph. From (12), y will be a maximum when

$$\cos(\pi t - 0.927) = 1,$$

that is,

$$\pi t - 0.927 = 0$$

$$t = \frac{0.927}{\pi} \approx 0.295,$$

and y will be a minimum when

$$\cos(\pi t - 0.927) = -1,$$

that is,

$$\pi t - 0.927 = \pi$$

$$t = 1 + \frac{0.927}{\pi} \approx 1.295,$$

where, in each case, we have taken the smallest positive value of t.

We now put the general equation

$$y = B \cos \omega t + C \sin \omega t \qquad (13)$$

in the form (1). If we proceed exactly as before, using (6) and (13), we find that for specified B and C, $A = \sqrt{B^2 + C^2}$ and a solution of the equations

$$\cos \alpha = \frac{B}{A} \quad \text{and} \quad \sin \alpha = \frac{C}{A} \qquad (14)$$

will determine a unique α in the interval from 0 to 2π, from which the form (1) follows. (See Exercise 3 below.)

Exercises 5-8

1. What is the smallest positive value of t for which the graph of Equation (4) crosses the t-axis? Compare your result with the data shown in Figure 5-4e.
2. Sketch each of the following graphs over at least two of its periods. Show the amplitude, period, and phase of each.
   a) $y = 2 \cos 3t$
   b) $y = 2 \cos(\frac{3t}{2})$
   c) $y = 3 \cos(-2t)$
   d) $y = -2 \sin(\frac{t}{3})$ (Remember that the phase is defined to be positive.)

e) $y = -2 \sin(2t + \pi)$

f) $y = 5 \cos(3t + \frac{\pi}{6})$

3. Express each of the following equations in the form $y = A \cos(\pi t - \alpha)$ for some appropriate real numbers A and α.
   a) $y = 4 \sin \pi t - 3 \cos \pi t$
   b) $y = -4 \sin \pi t + 3 \cos \pi t$
   c) $y = -4 \sin \pi t - 3 \cos \pi t$
   d) $y = 3 \sin \pi t + 4 \cos \pi t$
   e) $y = 3 \sin \pi t - 4 \cos \pi t$

4. Without actually computing the value of α, show on a diagram how A and α can be determined from the coefficients B and C of $\cos \omega t$ and $\sin \omega t$ if each of the following expressions of the form $B \cos \omega t + C \sin \omega t$ is made equal to $A \cos(\omega t - \alpha)$. Compute α, and find the maximum and minimum values of each expression, and its period. Give reasons for your answers.
   a) $3 \sin 2t + 4 \cos 2t$
   b) $2 \sin 3t - 3 \cos 3t$
   c) $-\sin(\frac{t}{2}) + \cos(\frac{t}{2})$

5. Verify that the superposition of any two pure waves $A \cos(\omega t - \alpha)$ and $B \cos(\omega t - \beta)$ is a pure wave of the same frequency, that is, that there exist real values C and γ such that

$$A \cos(\omega t - \alpha) + B \cos(\omega t - \beta) = C \cos(\omega t - \gamma).$$

6. Solve for all values of t:
   a) $3 \cos \pi t + 4 \sin \pi t = 2.5$

   (Method: From equation (12) we see that this equation is equivalent to $5 \cos(\pi t - 0.927) = 2.5$. For every solution, we have

$$\cos(\pi t - 0.927) = 0.5,$$

   which is satisfied only if the argument of the cosine is $\frac{\pi}{3} + 2n\pi$ or $-\frac{\pi}{3} + 2n\pi$. It follows that the equation is satisfied for all values of t such that

[sec. 5-8]

$\pi t - 0.927 = \pm\frac{\pi}{3} + 2n\pi$ or such that

$$t \approx \frac{0.927}{\pi} \pm \frac{1}{3} + 2n.$$

Question: What is the smallest positive value of t for which equation (a) is satisfied?)

b) $3 \cos \pi t + 4 \sin \pi t = 5$

c) $\sin 2t - \cos 2t = 1$

d) $4 \cos \pi t - 3 \sin \pi t = 0$

e) $4 \cos \pi t + 3 \sin \pi t = 1$

*7. Show that any wave of the form

$$y = B \cos(\mu t - \beta), \quad (\mu \neq 0),$$

can be written in the form (1), that is,

$$y = A \cos(\omega t - \alpha)$$

where A is non-negative, ω positive and $0 \leq \alpha < 2\pi$.

5-9. Identities

In the analysis of general periodic motions the product of two circular functions often appears, and the expression of a product as the sum or difference of two circular functions is quite useful. Such expressions can be derived by taking the sums and differences of the circular functions of $x + y$ and $x - y$. In fact we have

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)] \quad (1)$$

$$\sin x \sin y = -\tfrac{1}{2}[\cos(x + y) - \cos(x - y)] \quad (2)$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]. \quad (3)$$

One interesting fact about these product formulas is that they can be used to obtain formulas expressing sums as products. We merely set $x + y = \alpha$ and $x - y = \beta$ in Equations (1), (2) and (3). Observing that $x = \frac{\alpha + \beta}{2}$ and $y = \frac{\alpha - \beta}{2}$ we have

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad (4)$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \qquad (5)$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad (6)$$

Formulas (1), (2), (3), and (4), (5), (6) need not be memorized. The important thing is to know how to derive them. (See Exercise (1) below.)

It is often useful to have some expression involving circular functions in more than one form. That is, we sometimes wish to replace one expression by another expression to which it is equal for all values of the variable for which both expressions are defined. A statement of this kind of relationship between two expressions is called an identity. For example,

$$\sin^2 x = 1 - \cos^2 x$$

is an identity, because it is true for all real x. To show that a given equation is an identity, we try to transform one side into the other or both sides into identical expressions. As an example, consider the equation

$$\cos^3\theta + \sin^2\theta\cos\theta = \cos\theta.$$

We note that by factoring $\cos\theta$ from the terms on the left, we have

$$\cos\theta(\cos^2\theta + \sin^2\theta) = \cos\theta,$$

and since $\cos^2\theta + \sin^2\theta = 1$, we have

$$\cos\theta = \cos\theta,$$

and the identity is established.

Exercises 5-9

1. Derive formulas (1), (2) and (3) from the appropriate formulas in Section 5-6.
2. In formulas (1), (2) and (3) let $x = m\alpha$ and $y = n\alpha$ thus deriving formulas for
    a) $\cos m\alpha \cos n\alpha$,

b) $\sin m\alpha \sin n\alpha$,

c) $\sin m\alpha \cos n\alpha$.

3. Using (6) derive a formula for $\sin\alpha - \sin\beta$.

4. Using any of the formulas (4), (5), (6) derive a formula for $\cos x - \sin x$.

5. Using any of the formulas developed in this chapter, find:

   a) $\sin \frac{\pi}{12}$; (Hint: $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$)

   b) $\cos \frac{5\pi}{12}$;

   c) $\tan \frac{7\pi}{12}$;

   d) $\cos \frac{11\pi}{12}$.

6. Using any of the formulas developed in this chapter, show that for all values where the functions are defined the following are identities:

   a) $\cos^4\theta - \sin^4\theta = \cos 2\theta$;

   b) $\cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$;

   c) $\frac{1}{\cos^2\alpha} - 1 = \tan^2\alpha$;

   d) $\cos(\alpha - \pi) = \cos(\alpha + \pi)$;

   e) $\tan \frac{1}{2}\theta = \frac{\sin\theta}{1 + \cos\theta}$; (See Exercise 5-6, 11.)

   f) $\cos^2 \frac{1}{2}\theta = \frac{\tan\theta + \sin\theta}{2\tan\theta}$;

   g) $1 + \sin\alpha = (\sin\frac{1}{2}\alpha + \cos\frac{1}{2}\alpha)^2$;

   h) $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$;

   i) $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$;

   j) $\frac{1 + \cos\theta}{\sin\theta} + \frac{\sin\theta}{1 + \cos\theta} = \frac{2}{\sin\theta}$;

   k) $\frac{\sin 2\alpha}{\sin\alpha} - \frac{\cos 2\alpha}{\cos\alpha} = \frac{1}{\cos\alpha}$,

7. In Exercise 7 of Section 5-6 you derived the formula:

$$\cos 2x = 2\cos^2 x - 1.$$

a) Solve this for $\cos^2 x$ thus expressing $\cos^2 x$ as a linear function of $\cos 2x$.

b) Consider $\cos^4 x$ as $(\cos^2 x)^2$ and by the same methods as used in (a) show that

$$\cos^4 x = \frac{1}{8}(3 + 4 \cos 2x + \cos 4x).$$

8. Using the formula $\cos 2x = 1 - 2 \sin^2 x$ derive the formula for $\sin^4 x$:

$$\sin^4 x = \frac{1}{8}(3 - 4 \cos 2x + \cos 4x).$$

9. Show that the following are identities: that is, they are true for all values for which the functions are defined.

a) $\sin 2\theta \cos\theta - \cos 2\theta \sin\theta = \sin\theta$.

b) $\sin(x - y) \cos z + \sin(y - z) \cos x = \sin(x - z) \cos y$.

c) $\sin 3x \sin 2x = \frac{1}{2}(\cos x - \cos 5x)$.

d) $\cos\theta - \sin\theta \tan 2\theta = \dfrac{\cos 3\theta}{\cos 2\theta}$.

e) $\sin^3\theta = \frac{1}{4}(3 \sin\theta - \sin 3\theta)$.

f) $\sin x + \sin 2x + \sin 3x = \sin 2x(2 \cos x + 1)$.

g) $\left(\dfrac{1 + \tan x}{1 - \tan x}\right)^2 = \dfrac{1 + \sin 2x}{1 - \sin 2x}$.

5-10. Tangents at $x = 0$ to the Graphs of $y = \sin x$ and $y = \cos x$

a) The tangent T to the graph G of $\sin: x \longrightarrow \sin x$ at $P(0,0)$ turns out to be the line $y = x$, as we shall show. (See Figure 5-10a.) We shall at first consider only points on G which are to the right of P, that is, we shall take $x > 0$. We shall also assume that $x < \frac{\pi}{2}$; this will do no harm, since we are concerned only with the shape of G near P.

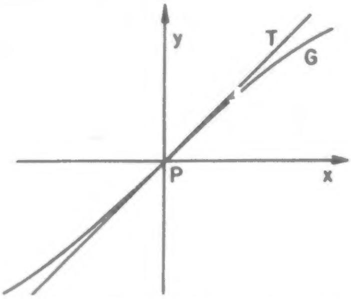

Figure 5-10a. Graph of $y = \sin x$ and $y = x$

In Figure 5-10b, which shows a portion of the unit circle, BC is perpendicular to OA and hence is shorter than the arc BA.

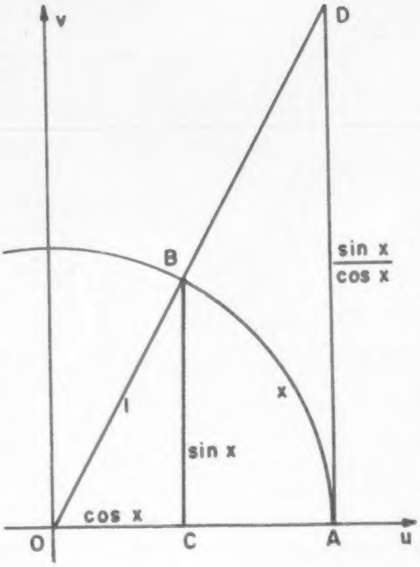

Figure 5-10b. Part of the unit circle.

But if the length of arc BA is x, then the lengths of CB and OC are sin x and cos x, respectively, and therefore

$$\sin x < x. \qquad (1)$$

This means that, in Figure 5-10a, G lies below the line $y = x$, to the right of P, as we have indicated.

In Figure 5-10b, AD has been drawn perpendicular to OA at A, meeting OB extended at D. By similar triangles,

$$\frac{AD}{OA} = \frac{CB}{OC},$$

and therefore

$$AD = \frac{\sin x}{\cos x}.$$

Now, the area of triangle OAD is

$$\frac{1}{2}(OA)(AD) = \frac{1}{2}(1)\left(\frac{\sin x}{\cos x}\right),$$

the area of sector OAB is

$$\frac{1}{2}(1)^2(x) = \frac{1}{2}x,$$

and the area of triangle OAD is greater than the area of sector OAB, or

$$\frac{1}{2}\,\frac{\sin x}{\cos x} > \frac{1}{2}x.$$

Because, cos x is positive for $0 < x < \frac{\pi}{2}$,

$$\sin x > x \cos x. \qquad (2)$$

Since

$$1 > \cos x$$

and, again, cos x is positive, we have by multiplication

$$\cos x > \cos^2 x = 1 - \sin^2 x.$$

Now, from (1), $\sin^2 x < x^2$ for $0 < x < \frac{\pi}{2}$,

so that
$$\cos x > 1 - x^2. \qquad (3)$$
It now follows from (2) that
$$\sin x > (1 - x^2)x,$$
and therefore, for all $x^2 < \epsilon$,
$$\sin x > (1 - \epsilon)x. \qquad (4)$$
Combining this result with (1), we conclude that, to the right of P, G lies between the lines $y = x$ and $y = (1 - \epsilon)x$ for x small enough. (See Figure 5-10c). Since G is symmetric with respect to the origin, it lies between the same lines on a corresponding interval to the left of P. Therefore, the line $y = x$ is the best linear approximation to G near P.

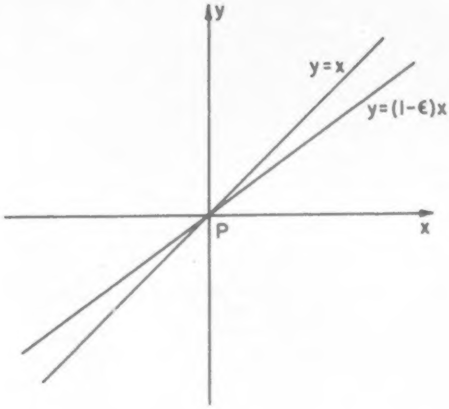

Figure 5-10c. Linear approximations to y = sin x.

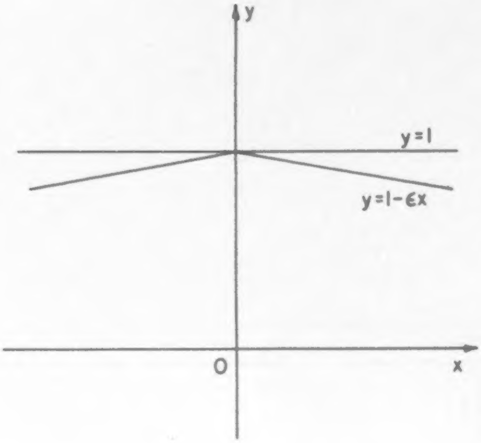

Figure 5-10d. Linear approximations to y = cos x.

b) Let us now turn to the graph of cos: $x \to \cos x$ near P(0,1), the point of intersection with the y-axis. Since the graph is symmetric with respect to the y-axis, it is sufficient to consider positive values of x. (See Figure 5-10d).

We know, for $0 < x < \frac{\pi}{2}$, that

$$\cos x < 1;$$

hence G lies below the line $y = 1$. By (3), moreover,

$$\cos x > 1 - x^2,$$

that is

$$\cos x > 1 + (-x)x,$$

or

$$\cos x > 1 - \epsilon x$$

for

$$0 < x < \epsilon .$$

Since ϵ can be chosen arbitrarily small, it follows that the tangent T is necessarily the line $y = 1$.

5-11. Tangent to the Graph of Sine or Cosine at a General Point

To find an equation of the tangents to the graphs of $x \rightarrow \sin x$ and of $x \rightarrow \cos x$ at a point where $x = h$, we must use the addition formulas. We write $x = h + (x - h)$. Then

$$\sin[h + (x - h)] = \sin h \cos(x - h) + \cos h \sin(x - h),$$

$$\cos[h + (x - h)] = \cos h \cos(x - h) - \sin h \sin(x - h).$$

We now replace $\cos(x - h)$ and $\sin(x - h)$ by their best linear approximations, namely by 1 and by $x - h$, respectively, and obtain for the required tangent lines

$$y = \sin h + (\cos h)(x - h) \qquad (1)$$

and

$$y = \cos h - (\sin h)(x - h). \qquad (2)$$

According to (1), the slope of the line tangent at $(h, \sin h)$ to the graph of the function $\sin: x \rightarrow \sin x$ is $\cos h$. Hence the associated slope function is $\cos: x \rightarrow \cos x$ or

$$\sin' = \cos.$$

Similarly, from (2), the slope of the line tangent at $(h, \cos h)$ to the graph of the function cos is $-\sin h$. Hence

$$\cos' = -\sin.$$

Exercises 5-11

1. Write an equation of the line tangent to $y = \sin x$ at the point where
    a) $x = \frac{\pi}{3}$
    b) $x = \frac{4\pi}{3}$
    c) $x = 1$.
2. Write an equation of the line tangent to $y = \cos x$ at the point where
    a) $x = \frac{\pi}{6}$
    b) $x = 2$.
3. What is the error involved in using x as an approximation to $\sin x$ when
    a) $x = 0$
    b) $x = .1$
    c) $x = .2$
    d) $x = .3$?

5-12. Analysis of General Waves

In Sections 5-4 and 5-8 we considered the superposition of two pure waves of the same period (or frequency). We found that the superposition of such waves is again a pure wave of the given frequency. Next we ask what conclusion we can draw about the superposition of two waves with different periods. Suppose, for example, that we had to deal with

$$y = 2 \sin 3x - 3 \cos 2x.$$

Unfortunately, $\sin 3x$ and $\cos 2x$ have different fundamental periods, $\frac{2\pi}{3}$ and π, so they cannot be combined into a single term, the way we could if we had only $\cos 3x$ and $\sin 3x$, say, or $\cos 2x$ and $\sin 2x$. However, any multiple of a period can be looked upon as a period. That is, we can consider $y = 2 \sin 3x$ as having a period of $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, 2π, $\frac{8\pi}{3}$, or any other integral multiple of $\frac{2\pi}{3}$. Similarly, $y = 3 \cos 2x$ can be considered as having a period of π, 2π, 3π, etc. Now comparing these values, we note that both expressions can be considered as having a period of 2π, and hence their difference will also have a period of 2π. In effect, we simply find the least common multiple of the periods of two dissimilar expressions of this form and we have the period of their sum or difference. There is little else that we can conclude in general. About all we can do to simplify matters is to sketch separately the graphs of

$$u = 2 \sin 3x, \quad v = 3 \cos 2x,$$

and $y = u - v$. The result is shown by the three curves in Figure 5-12a.

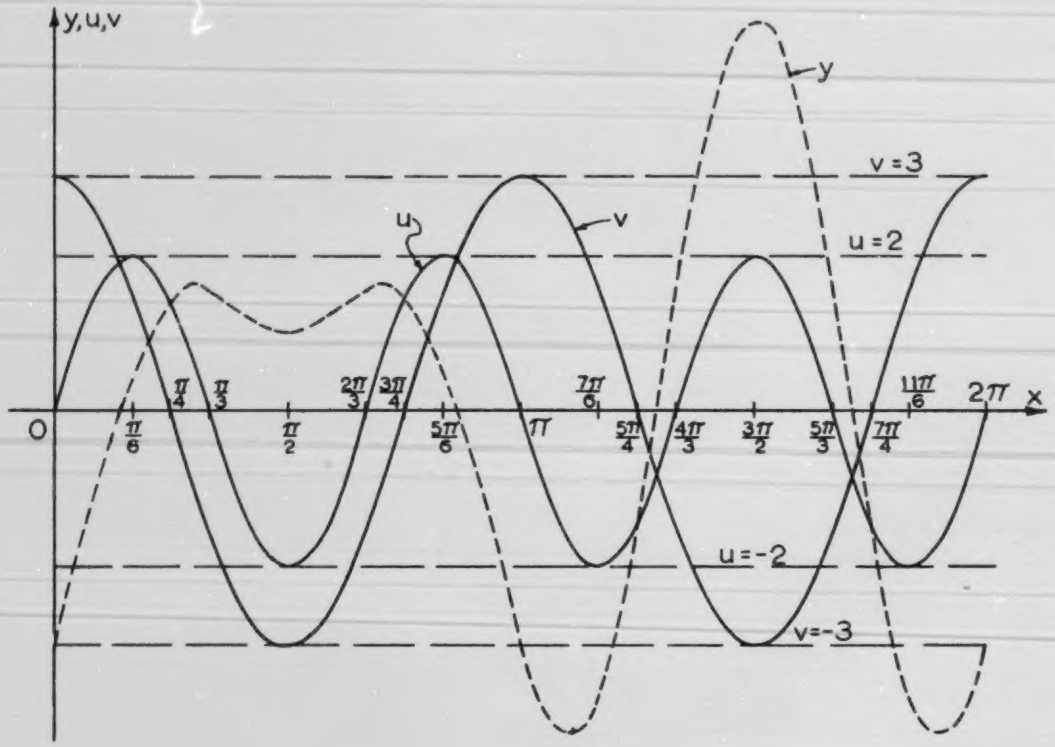

Figure 5-12a. $u = 2 \sin 3x$, $v = 3 \cos 2x$

$y = u - v = 2 \sin 3x - 3 \cos 2x$,

$0 \leq x \leq 2\pi$.

The superposition of sine and cosine waves of different periods can produce quite complicated curves. In fact, with only slight restrictions, any periodic function can be approximated arbitrarily closely as a sum of a finite number of sines and cosines. The subject of harmonic analysis or Fourier series is concerned with approximating periodic functions in this way. The principal theorem, first stated by Fourier, is that a function f of period a can be approximated arbitrarily closely by sines and cosines for each of which some multiple of the fundamental period is a. Specifically,

$$\begin{aligned} f(x) \approx A_0 &+ \left(A_1 \cos \frac{2\pi x}{a} + B_1 \sin \frac{2\pi x}{a}\right) \\ &+ \left(A_2 \cos \frac{4\pi x}{a} + B_2 \sin \frac{4\pi x}{a}\right) \\ &+ \ldots. \qquad (1) \\ &+ \left(A_n \cos \frac{2n\pi x}{a} + B_n \cos \frac{2n\pi x}{a}\right), \end{aligned}$$

and the more terms we use, the better is our approximation.

As an example, consider the function depicted in Figure 5-12b. This function is defined on the interval $-\pi \leq x < \pi$ by

$$f(x) = \begin{cases} 0, & \text{if } x = -\pi \\ -1, & \text{if } -\pi < x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x < \pi. \end{cases} \qquad (2)$$

For all other values of x we define $f(x)$ by the periodicity condition

$$f(x + 2\pi) = f(x).$$

This function has a particularly simple approximation as a series of the form (1), namely,

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n - 1)x}{2n - 1}\right). \qquad (3)$$

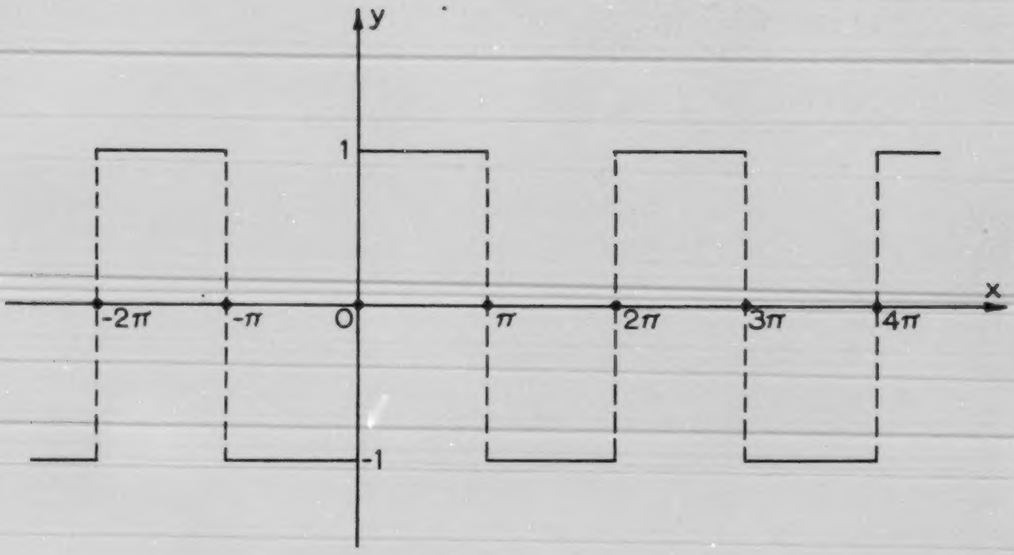

Figure 5-12b. Graph of periodic function.

$$x \to f(x) = \begin{cases} 0, & \text{if } x = -\pi \\ 1, & \text{if } 0 < x < \pi \\ 0, & \text{if } x = 0 \\ -1, & \text{if } -\pi < x < 0 \end{cases} ; \quad f(x + 2\pi) = f(x).$$

Fourier series: $\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ . \ . \ . + \frac{\sin(2n - 1)x}{2n - 1}\right).$

As an exercise, you may graph the successive approximations to $f(x)$ by taking one, then two, then three terms of the series, and see how the successive graphs approach the graph of $y = f(x)$.

The problem of finding the series (1) for any given periodic function f is taken up in calculus.

Exercises 5-12

1. Sketch graphs, for $|x| < \pi$, for each of the following curves.

   a) $y = \frac{4}{\pi} \sin x.$

b) $y = \frac{4}{\pi}(\frac{\sin x}{1} + \frac{\sin 3x}{3})$.

c) $y = \frac{4}{\pi}(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5})$.

2. a) Find the periods of each of the successive terms of the series (3), namely,

$$\sin x, \frac{\sin 3x}{3}, \frac{\sin 5x}{5}, \ldots$$

b) What terms of the general series (1) are missing? From the symmetry properties of the function f defined by (2) can you see a reason for the absence of certain terms?

5-13. Inverse Circular Functions and Trigonometric Equations

We have now reached the point in our study of the circular functions where we might well ask if there exist inverse functions for them. The necessary and sufficient condition developed in Section 4-9 for a function f to have an inverse is that it be one-to-one. In other words for every 2 different numbers x_1 and x_2 in the domain the values of the function must be different. That is, if x_1 and x_2 are in the domain of f, and $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. It is obvious that this condition is not satisfied for sine, cosine, and tangent since these functions are periodic. For example we know that: $\sin 0 = \sin 2\pi$, $\cos \frac{\pi}{4} = \cos \frac{9\pi}{4}$, and $\tan \frac{\pi}{3} = \tan \frac{4\pi}{3}$, etc. It follows that there can be no inverse functions for sine, cosine or tangent.

Suppose, however, we restrict the domain of sin to

$$\{x: -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}.$$

Since $\sin x_1 < \sin x_2$ if $-\frac{\pi}{2} \leq x_1 < x_2 \leq \frac{\pi}{2}$, it follows that the sine function, thus restricted, is strictly increasing and by Corollary 4-2-1 therefore has an inverse which we call $\sin^{-1}$. (See Figures 5-13a and 5-13b.)

By imposing different restrictions on the domain of the sine function (for example, by choosing as the domain the set of real numbers x such that $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$) we may again obtain a one-to-one function which has an inverse. However, the function $x \rightarrow \sin x$, where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, gives rise to what is called the principal inverse sine function and when we speak of the inverse sine it is this one that we mean.

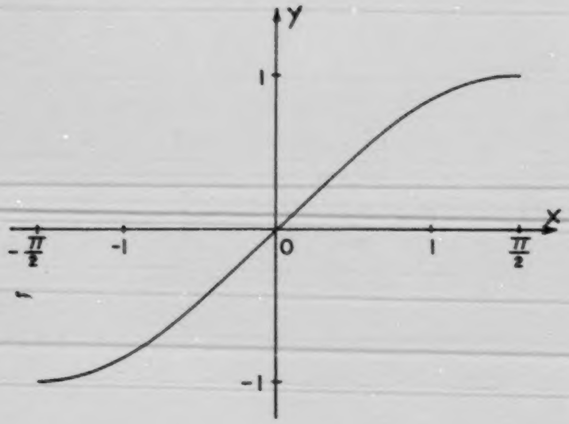

Figure 5-13a. Graph of
$y = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

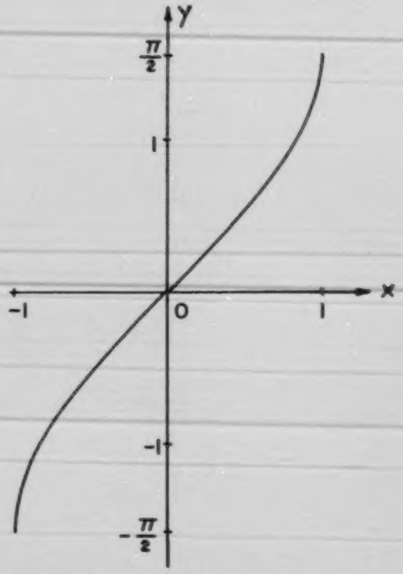

Figure 5-13b. Graph of
$y = \sin^{-1}x, -1 \leq x \leq 1$.

Fortunately there is universal agreement among mathematicians on the definition that we have given.

In like manner if we suitably restrict the domain of cosine, we obtain a function which is one-to-one and therefore has an inverse. Again there are different possibilities and we choose the restriction $0 \leq x \leq \pi$, and call the inverse, $\cos^{-1}$.

The domain of both $\sin^{-1}$ and $\cos^{-1}$ is the interval $[-1, +1]$ but the ranges are different; that of $\sin^{-1}$ being the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ while that of $\cos^{-1}$ is the interval $[0, \pi]$.

Example 1. Find $\sin^{-1} 0.5$.

Solution. We know that $\sin[\sin^{-1} 0.5] = 0.5$. We want that number in the restricted region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ such that $\sin x = 0.5$. It is of course $\frac{\pi}{6}$.

Example 2. Find $\cos^{-1}(-0.4)$.

Solution. $\cos(\cos^{-1}(-0.4)) = -0.4$. We want the number x, $0 \leq x \leq \pi$, such that $\cos x = -0.4$. Using Table I we find that $x = 1.98$.

Example 3. Find $\sin^{-1}(\sin \pi)$.

Solution. You might be tempted to say π but you should note that $\sin \pi = 0$ hence $\sin^{-1}(\sin \pi) = \sin^{-1} 0 = 0$. What this example clearly shows is that $\sin^{-1}$ and sin are not inverse functions unless the domain of sin is restricted to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ as we required to begin with.

Example 4. Find $\cos(\sin^{-1}(-\frac{\sqrt{3}}{2}))$.

Solution. We find that $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$. Since $\cos(-\frac{\pi}{3}) = +0.5$, $\cos(\sin^{-1}(-\frac{\sqrt{3}}{2})) = +0.5$.

To consider the problem of finding an inverse of the tangent function we must determine if there is a restricted domain where it is a strictly increasing or strictly decreasing function. Now the tangent is defined as follows:

$$\tan: x \longrightarrow \frac{\sin x}{\cos x}.$$

Since division by 0 is not defined, the domain of tan excludes all zeros of cos; these are the odd multiples of $\frac{\pi}{2}$.

Because the functions sin and cos have period 2π, it follows that this is also a period of tan, but, as we shall see, it is not the fundamental period. If a is any period of tan, then it must be true, for all x in the domain of tan, that

$$\tan(x + a) = \tan x,$$

or, using the definition of tan,

$$\frac{\sin(x + a)}{\cos(x + a)} = \frac{\sin x}{\cos x}.$$

Clearing of fractions and rearranging, we have

$$\sin(x + a) \cos x - \cos(x + a) \sin x = 0. \qquad (1)$$

The left-hand member of (1) is precisely Equation (4) of Section 5-7, with $\alpha = x + a$ and $\beta = x$; hence (1) becomes

$$\sin\big((x + a) - x\big) = 0,$$

or

$$\sin a = 0.$$

Hence a may be any multiple of π, and the smallest positive one of these, and therefore the fundamental period of tan, is π itself.

We wish now to show that tan is a strictly increasing function over the interval $\{x: -\frac{\pi}{2} < x < \frac{\pi}{2}\}$. First of all, over the non-negative portion of this interval, namely $\{x: 0 \leq x < \frac{\pi}{2}\}$, sin is increasing and cos is decreasing, that is, if

$$0 \leq x_1 < x_2 < \frac{\pi}{2},$$

then

$$\sin x_1 < \sin x_2 \quad \text{and} \quad \cos x_1 > \cos x_2.$$

Hence

$$\tan x_1 = \frac{\sin x_1}{\cos x_1} < \frac{\sin x_2}{\cos x_1} < \frac{\sin x_2}{\cos x_2} = \tan x_2,$$

and tan is therefore strictly increasing over this interval.

But

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x,$$

which shows that the graph of tan is symmetric with respect to the origin. Hence we conclude that tan is also strictly increasing over $\{x: -\frac{\pi}{2} < x \leq 0\}$ and therefore over the entire interval $\{x: -\frac{\pi}{2} < x < \frac{\pi}{2}\}$. It follows from Corollary 4-2-1 that, over this interval, tan has an inverse $\tan^{-1}$. We can draw the graph of $y = \tan x$ in this region by considering a table of values and the behavior of $\sin x$ and $\cos x$.

x	sin x	cos x	tan x = $\frac{\sin x}{\cos x}$
$-\pi/2$	-1	0	undefined
$-\pi/3$	-0.87	0.5	-1.7
$-\pi/4$	-0.71	0.71	-1
$-\pi/6$	-0.5	0.87	-0.58
0	0	1	0
$\pi/6$	0.5	0.87	0.58
$\pi/4$	0.71	0.71	1
$\pi/3$	0.87	0.5	1.7
$\pi/2$	1	0	undefined

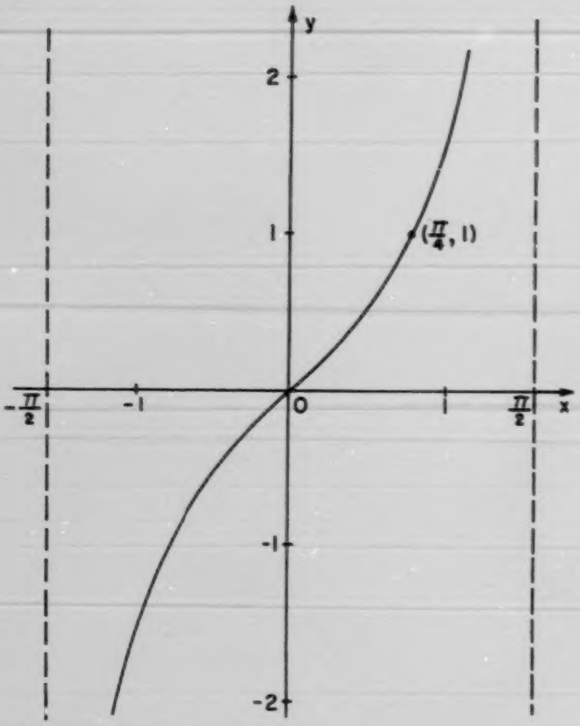

Figure 5-13c. Graph of $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The graph of $y = \tan^{-1} x$ is found by reflecting the graph of $y = \tan x$ in the line $y = x$.

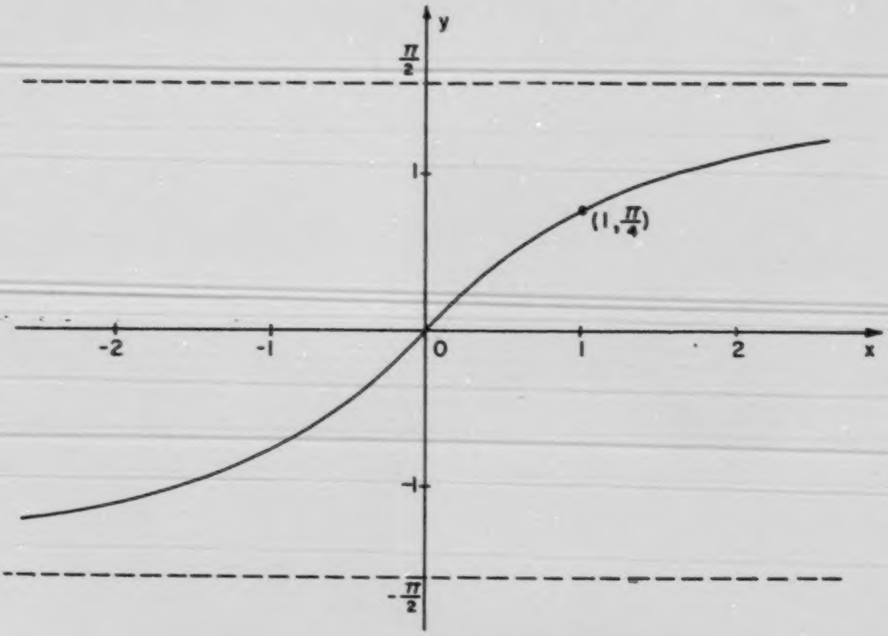

Figure 5-13d. Graph of $y = \tan^{-1}x$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

The inverse trigonometric functions will be very useful if you study the integral calculus later on. They can also be used to express the solutions to many trigonometric equations, much as radicals can be used to express the solutions to many algebraic equations.

Trigonometric Equations. We have solved some trigonometric equations before. We shall here do a few more. In solving an equation we are as always looking for the set of all those numbers which make the given statement true.

Example 5. Solve $\sin x = \frac{1}{2}$.

Solution. One number in the solution set is $\sin^{-1} 0.5$ which we know is $\frac{\pi}{6}$. Are there any others? Because the sine is periodic with period 2π we know that all numbers of the form $\frac{\pi}{6} + 2n\pi$

belong to the solution set. Now in the region from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, the sine is a strictly increasing one-to-one function and thus can take on each value in its range only once, but from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ it is strictly decreasing and again one-to-one so it takes on every value once more. We know that $\sin(\pi - x) = \sin x$ and therefore $\pi - \frac{\pi}{6}$ or $\frac{5\pi}{6}$ is the only other number in the interval $\{x: 0 \leq x < 2\pi\}$ which satisfies the equation. The complete solution set of possible values then consists of

$$\frac{\pi}{6} + 2n\pi \quad \text{and} \quad \frac{5\pi}{6} + 2n\pi.$$

Testing in the original equation, we find that they all check.

Example 6. Solve $\sin x + \cos x = 1$.

Solution. $\sin x + \cos x = 1$

$$\sin x = 1 - \cos x$$

Squaring, substituting for $\sin^2 x$ its value in terms of $\cos^2 x$, collecting and factoring, we get

$$1 - \cos x = 0 \quad \text{or} \quad 2 \cos x = 0.$$

Hence

$$\cos x = 1 \quad \text{or} \quad \cos x = 0$$

Using the periodicity we then get

$$x = 0 + 2n\pi \quad \text{or} \quad x = \frac{\pi}{2} + 2n\pi \quad \text{or} \quad -\frac{\pi}{2} + 2n\pi.$$

These are possible solutions of the original equation but we must test them to be sure.

$\sin 2n\pi + \cos 2n\pi = 0 + 1 = 1$ Check.

$\sin(\frac{\pi}{2} + 2n\pi) + \cos(\frac{\pi}{2} + 2n\pi) = 1 + 0 = 1$ Check.

$\sin(-\frac{\pi}{2} + 2n\pi) + \cos(-\frac{\pi}{2} + 2n\pi) = -1 + 0 \neq 1$. Fails to check.

Therefore the solution set is $\{2n\pi, \frac{\pi}{2} + 2n\pi\}$.

Example 7. Solve $6 \cos^2 x + 5 \sin x = 0$.

Solution. We substitute $\cos^2 x = 1 - \sin^2 x$ to get an equation in $\sin x$.

$$6(1 - \sin^2 x) + 5 \sin x = 0$$

$$6 - 6 \sin^2 x + 5 \sin x = 0$$

$$6 \sin^2 x - 5 \sin x - 6 = 0$$

$$(3 \sin x + 2)(2 \sin x - 3) = 0$$

$$\sin x = -\frac{2}{3} \quad \text{or} \quad \sin x = \frac{3}{2}$$

But $\frac{3}{2}$ is not in the range of sine so there are no values of x which satisfy the second equation. The first equation yields one number: $x = \sin^{-1}(-\frac{2}{3})$ and the complete solution set is $\{\sin^{-1}(-\frac{2}{3}) + 2n\pi, \pi - \sin^{-1}(-\frac{2}{3}) + 2n\pi\}$. Testing we find that these all check.

Exercises 5-13

1. Sketch the graph of $y = \cos^{-1}x$, indicating clearly the domain and range of the function.
2. Evaluate
   a) $\sin^{-1}(-\frac{1}{2})$
   c) $\tan^{-1}(-\sqrt{3})$
   b) $\cos^{-1}(\frac{\sqrt{3}}{2})$
   d) $\cos^{-1} 1 - \sin^{-1}(-1)$
3. Find:
   a) $\sin(\cos^{-1} 0.73)$;
   c) $\sin[\cos^{-1} \frac{3}{5} + \sin^{-1}(-\frac{3}{5})]$;
   b) $\cos(\sin^{-1}(-0.47))$;
   d) $\sin[2 \cos^{-1} \frac{5}{13}]$.
4. a) Show that $\sin(\cos^{-1} \frac{2}{3}) = \cos(\sin^{-1} \frac{2}{3})$.

   b) Is it true that for all x, $\sin(\cos^{-1}x) = \cos(\sin^{-1}x)$?
5. Show that $\sin(\tan^{-1}x) = +\frac{x}{\sqrt{1 + x^2}}$. Why is the sign + rather than - or $\pm$?
6. Express in terms of x:
   a) $\sin(2 \tan^{-1}x)$;
   c) $\tan(\cos^{-1}x)$;
   b) $\tan(2 \tan^{-1}x)$;
   d) $\sin[\sin^{-1}x + \cos^{-1}x]$.

In the solution of the following equations be sure that you have (a) not lost any true solutions or, (b) introduced any numbers as solutions which do not satisfy the original equations.

7. Solve for x:

a) $\sin x + \cos x = 0$; c) $3 \tan x - \sqrt{3} = 0$;

b) $4 \cos^2 x - 1 = 0$; d) $4 \tan x + \sin 2x = 0$.

8. Solve for x:

a) $2 \cos x - \sin x = 1$; c) $\tan x = \frac{1}{\tan x}$;

b) $9 \cos^2 x + 6 \cos x = 8$; d) $\cos 2x - 1 = \sin x$.

9. Solve for x:

a) $2 \sin^{-1} x = \frac{\pi}{4}$; c) $2 \sin^{-1} 2x = 3$;

b) $\sin 2x = \cos(\pi - x)$; d) $3 \sin 2x = 2$.

*10. It sometimes happens that you want to solve an equation of the form $x = \tan x$ or $x \cdot 2^x = 2$, or $x + 2 \sin x = 0$. No methods we have developed so far seem to do this. However, our present knowledge of graphing functions enables us to get at least approximate solutions of these equations.

We put the given equation in the form

$$f(x) = g(x)$$

where f and g are functions whose graphs are familiar. The points of intersection of the graphs will give values of x which satisfy the original equation.

a) Solve: $x = \tan x$.

b) Solve: $x \cdot 2^x = 2$ by first setting $2^x = \frac{2}{x}$ and graphing $y = 2^x$ and $y = \frac{2}{x}$.

c) Solve: $x + 2 \sin x = 0$.

d) Solve: $x = \sin^{-1} x$.

e) Solve: $\sin x = e^{-x}$.

5-14. Summary of Chapter 5

We define the circular functions cos and sin as follows: if the points $P(u, v)$ and $A(1, 0)$ are on the unit circle, and if the counterclockwise arc AP is x units long, then $u = \cos x$ and $v = \sin x$.

A function f is periodic, with period a, if, for each x in the domain of f, $x + a$ is also in the domain of f, and $f(x + a) = f(x)$. The smallest positive a which satisfies this relation is the fundamental period of f. The fundamental period of both cos and sin is 2π.

We define $\tan: x \rightarrow \frac{\sin x}{\cos x}$, with fundamental period π.

Radian and degree measure of angles are defined. They are related by the formula π radians $= 180^\circ$.

We summarize some of the properties of a class of plane vectors and define a class of functions from vectors to vectors, called rotations. These functions are used to derive the formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

A pure wave is defined to have the form $y = A\cos(\omega t - \alpha)$, $A > 0$, $\omega > 0$, $0 \leq \alpha < 2\pi$. The number α is the phase of the wave; a sine wave has phase $\frac{\pi}{2}$. The sum of two waves of a given period is a wave of the same period; in particular, if $u = B\cos\omega t$ and $v = C\sin\omega t$, then $u + v = A\cos(\omega t - \alpha)$, where $A = \sqrt{B^2 + C^2}$, $\cos\alpha = \frac{B}{A}$, and $\sin\alpha = \frac{C}{A}$.

The sum of two waves of commensurable periods is a wave whose fundamental period is the least common multiple of their fundamental periods. A very general class of periodic functions can be approximated with arbitrary accuracy by a finite sum of cosines and sines.

The slope functions associated with cos and sin are, respectively, $-\sin$ and cos: $\cos' = -\sin$, $\sin' = \cos$.

Miscellaneous Exercises

1. Determine whether each of the following functions is periodic; if a function is periodic, determine its fundamental period.
   a) $y = |\sin x|$.
   b) $y = x - [x]$, where $[x]$ is the greatest integer n such that $n \leq x$.
   c) $y = x \sin x$.
   d) $y = \sin^2 x$.
   e) $y = \sin(x^2)$.
   f) $y = \dfrac{\sin x + 2 \cos x}{2 \sin x + \cos x}$.
   g) $y = \sin x + |\sin x|$.
   *h) $y = \sin x + \sin(\sqrt{2}x)$

2. Consider the function $f: x \longrightarrow f(x)$ whose domain is the set of positive integers and for which $f(x)$ is the integer in the x-th decimal place in the decimal expansion of $\frac{110}{909}$. What is the range of f? Is f periodic? If so, what is its fundamental period? Find $f(97)$.

3. Given a function $f: x \longrightarrow f(x)$, with the properties $f(x + 2) = f(x)$, $f(-x) = -f(x)$, and $f(\frac{1}{2}) = 3$, evaluate the following:
   a) $f(\frac{9}{2})$
   b) $f(\frac{7}{2})$
   c) $f(9) + f(-7)$

4. Change the following angles to degrees:
   a) $\frac{22}{7}$ radians
   b) $\frac{2}{\pi}$ radian

5. Change the following angles to radians:
   a) 87°
   c) $\frac{\pi}{2}$ degrees
   b) $\frac{2}{\pi}$ degrees

6. What is the area of a sector with perimeter c and central angle k?

7. Find the fundamental period, amplitude, and range of each of the following curves. Sketch the curve over one cycle.

   a) $y = 2 \sin 3x$.

   b) $y = -3 \sin 2\pi x$.

   c) $y = 2 \cos \frac{x}{2\pi}$.

   d) $y = 6 \sin x \cos x$.

   e) $y = \sqrt{3} \sin 2x + \cos 2x$.

8. Determine A, B, and C so that the function
$f: x \longrightarrow 3 \cos(2\pi x + \frac{\pi}{2})$ can be described as
$f: x \longrightarrow A \sin(Bx + C)$.

9. If a condenser with a capacitance of C farads and containing a charge of Q_0 coulombs is placed in series with a coil of negligible resistance and an inductance of L henrys, the charge Q on the condenser t seconds later is given by

$$Q = Q_0 \sin\left(\frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right).$$

If $L = 0.4$ henry, and $C = 10^{-5}$ farad, find

   a) the fundamental frequency of this circuit;

   b) the time t_0 when $Q = 0$ for the first time;

   c) the time t_1 when $Q = 0.5\, Q_0$ for the first time;

   d) the time t_2 when $Q = 0.5\, Q_0$ for the second time.

10. Show that

$$\tan \frac{3x}{2} = \frac{\cos x - \cos 2x}{\sin 2x - \sin x}$$

for all values of x for which $\tan \frac{3x}{2}$ is defined.

11. Sketch the graphs of the following:

   a) $\cos x + |\cos x|$.

   b) $|\sin x| + \sin 2x$.

*12. Prove the following: (Hint: the formula for $\sin(x + y)$ is needed.)

   a) $|\sin x + \cos x| \leq \sqrt{2}$. b) $|\sqrt{3} \sin x + \cos x| \leq 2$.

*13. Find addition formulas for the following functions; that is, express $f(x + y)$ in terms of $f(x)$ and $f(y)$.

a) $f: x \longrightarrow f(x) = 2x + 3$.

b) $f: x \longrightarrow f(x) = \frac{2x + 3}{x + 4}$. (Hint: what is x in terms of $f(x)$?)

c) $f: x \longrightarrow 2(\cos x + i \sin x)$.

**14. Is there an x between 0 and 2π such that

$$\sin(\cos x) = \cos(\sin x)?$$

15. Sketch the graphs of the following, grouping together those with the same graphs.

a) $y = \sin x$.

b) $y = \sqrt{1 - \cos^2 x}$.

c) $y = |\sin x|$.

d) $y^2 = \sin^2 x$.

e) $y^2 = 1 - \cos^2 x$.

f) $y = \sqrt{\frac{1 - \cos 2x}{2}}$.

g) $y^2 = \frac{1 - \cos 2x}{2}$.

h) $y = 2 \sin \frac{x}{2} \cos \frac{x}{2}$.

i) $y = 2|\sin \frac{x}{2}| \cos \frac{x}{2}$.

j) $y = 2 \sin \frac{x}{2} |\cos \frac{x}{2}|$.

k) $y = 2|\sin \frac{x}{2} \cos \frac{x}{2}|$.

l) $y^2 = 4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}$.

Table I

Values of sin x and cos x for $0 \leq x \leq 1.57$.

x	sin x	cos x	x	sin x	cos x
.00	.0000	1.0000	.40	.3894	.9211
.01	.0100	1.0000	.41	.3986	.9171
.02	.0200	.9998	.42	.4078	.9131
.03	.0300	.9996	.43	.4169	.9090
.04	.0400	.9992	.44	.4259	.9048
.05	.0500	.9988	.45	.4350	.9004
.06	.0600	.9982	.46	.4439	.8961
.07	.0699	.9976	.47	.4529	.8916
.08	.0799	.9968	.48	.4618	.8870
.09	.0899	.9960	.49	.4706	.8823
.10	.0998	.9950	.50	.4794	.8776
.11	.1098	.9940	.51	.4882	.8727
.12	.1197	.9928	.52	.4969	.8678
.13	.1296	.9916	.53	.5055	.8628
.14	.1395	.9902	.54	.5141	.8577
.15	.1494	.9888	.55	.5227	.8525
.16	.1593	.9872	.56	.5312	.8473
.17	.1692	.9856	.57	.5396	.8419
.18	.1790	.9838	.58	.5480	.8365
.19	.1889	.9820	.59	.5564	.8309
.20	.1987	.9801	.60	.5646	.8253
.21	.2085	.9780	.61	.5729	.8196
.22	.2182	.9759	.62	.5810	.8139
.23	.2280	.9737	.63	.5891	.8080
.24	.2377	.9713	.64	.5972	.8021
.25	.2474	.9689	.65	.6052	.7961
.26	.2571	.9664	.66	.6131	.7900
.27	.2667	.9638	.67	.6210	.7838
.28	.2764	.9611	.68	.6288	.7776
.29	.2860	.9582	.69	.6365	.7712
.30	.2955	.9553	.70	.6442	.7648
.31	.3051	.9523	.71	.6518	.7584
.32	.3146	.9492	.72	.6594	.7518
.33	.3240	.9460	.73	.6669	.7452
.34	.3335	.9428	.74	.6743	.7385
.35	.3429	.9394	.75	.6816	.7317
.36	.3523	.9359	.76	.6889	.7248
.37	.3616	.9323	.77	.6961	.7179
.38	.3709	.9287	.78	.7033	.7109
.39	.3802	.9249	.79	.7104	.7038

Table I - Cont.

x	sin x	cos x
.80	.7174	.6967
.81	.7243	.6895
.82	.7311	.6822
.83	.7379	.6749
.84	.7446	.6675
.85	.7513	.6600
.86	.7578	.6524
.87	.7643	.6448
.88	.7707	.6372
.89	.7771	.6294
.90	.7833	.6216
.91	.7895	.6137
.92	.7956	.6058
.93	.8016	.5978
.94	.8076	.5898
.95	.8134	.5817
.96	.8192	.5735
.97	.8249	.5653
.98	.8305	.5570
.99	.8360	.5487
1.00	.8415	.5403
1.01	.8468	.5319
1.02	.8521	.5234
1.03	.8573	.5148
1.04	.8624	.5062
1.05	.8674	.4976
1.06	.8724	.4889
1.07	.8772	.4801
1.08	.8820	.4713
1.09	.8866	.4625
1.10	.8912	.4536
1.11	.8957	.4447
1.12	.9001	.4357
1.13	.9044	.4267
1.14	.9086	.4176
1.15	.9128	.4085
1.16	.9168	.3993
1.17	.9208	.3902
1.18	.9246	.3809
1.19	.9284	.3717
1.20	.9320	.3624
1.21	.9356	.3530
1.22	.9391	.3436
1.23	.9425	.3342
1.24	.9458	.3248
1.25	.9490	.3153
1.26	.9521	.3058
1.27	.9551	.2963
1.28	.9580	.2867
1.29	.9608	.2771
1.30	.9636	.2675
1.31	.9662	.2579
1.32	.9687	.2482
1.33	.9711	.2385
1.34	.9735	.2288
1.35	.9757	.2190
1.36	.9779	.2092
1.37	.9799	.1994
1.38	.9819	.1896
1.39	.9837	.1798
1.40	.9854	.1700
1.41	.9871	.1601
1.42	.9887	.1502
1.43	.9901	.1403
1.44	.9915	.1304
1.45	.9927	.1205
1.46	.9939	.1106
1.47	.9949	.1006
1.48	.9959	.0907
1.49	.9967	.0807
1.50	.9975	.0707
1.51	.9982	.0608
1.52	.9987	.0508
1.53	.9992	.0408
1.54	.9995	.0308
1.55	.9998	.0208
1.56	.9999	.0108
1.57	1.0000	.0008

Table II

Tables of sin and cos in decimal fractions of $\frac{\pi}{2}$.

x	sin x $\frac{\pi}{2}$	cos x $\frac{\pi}{2}$	
.00	.000	1.000	1.00
.01	.016	1.000	.99
.02	.031	1.000	.98
.03	.048	.999	.97
.04	.063	.998	.96
.05	.078	.997	.95
.06	.094	.996	.94
.07	.110	.994	.93
.08	.125	.992	.92
.09	.141	.990	.91
.10	.156	.988	.90
.11	.172	.985	.89
.12	.187	.982	.88
.13	.203	.979	.87
.14	.218	.976	.86
.15	.233	.972	.85
.16	.249	.969	.84
.17	.264	.965	.83
.18	.279	.960	.82
.19	.294	.956	.81
.20	.309	.951	.80
.21	.324	.946	.79
.22	.339	.941	.78
.23	.353	.935	.77
.24	.368	.930	.76
.25	.383	.924	.75
.26	.397	.918	.74
.27	.412	.911	.73
.28	.426	.905	.72
.29	.440	.898	.71
.30	.454	.891	.70
	cos y $\frac{\pi}{2}$	sin y $\frac{\pi}{2}$	y

Table II - Cont.

x	sin x $\frac{\pi}{2}$	cos x $\frac{\pi}{2}$	
.30	.454	.891	.70
.31	.468	.884	.69
.32	.482	.876	.68
.33	.495	.869	.67
.34	.509	.861	.66
.35	.523	.853	.65
.36	.536	.844	.64
.37	.549	.836	.63
.38	.562	.827	.62
.39	.575	.818	.61
.40	.588	.809	.60
.41	.600	.800	.59
.42	.613	.790	.58
.43	.625	.780	.57
.44	.637	.771	.56
.45	.649	.760	.55
.46	.661	.750	.54
.47	.673	.740	.53
.48	.685	.729	.52
.49	.696	.718	.51
.50	.707	.707	.50
	cos y $\frac{\pi}{2}$	sin y $\frac{\pi}{2}$	y

Table III

x°	$\sin^{\circ}x$	$\cos^{\circ}x$	$\tan^{\circ}x$	x°	$\sin^{\circ}x$	$\cos^{\circ}x$	$\tan^{\circ}x$
0	0.000	1.000	0.000				
1	.018	1.000	.018	46	0.719	0.695	1.036
2	.035	0.999	.035	47	.731	.682	1.072
3	.052	.999	.052	48	.743	.669	1.111
4	.070	.998	.070	49	.755	.656	1.150
5	.087	.996	.088	50	.766	.643	1.192
6	.105	.995	.105	51	.777	.629	1.235
7	.122	.993	.123	52	.788	.616	1.280
8	.139	.990	.141	53	.799	.602	1.327
9	.156	.988	.158	54	.809	.588	1.376
10	.174	.985	.176	55	.819	.574	1.428
11	.191	.982	.194	56	.829	.559	1.483
12	.208	.978	.213	57	.839	.545	1.540
13	.225	.974	.231	58	.848	.530	1.600
14	.242	.970	.249	59	.857	.515	1.664
15	.259	.966	.268	60	.866	.500	1.732
16	.276	.961	.287	61	.875	.485	1.804
17	.292	.956	.306	62	.883	.470	1.881
18	.309	.951	.325	63	.891	.454	1.963
19	.326	.946	.344	64	.899	.438	2.050
20	.342	.940	.364	65	.906	.423	2.145
21	.358	.934	.384	66	.914	.407	2.246
22	.375	.927	.404	67	.921	.391	2.356
23	.391	.921	.425	68	.927	.375	2.475
24	.407	.914	.445	69	.934	.358	2.605
25	.423	.906	.466	70	.940	.342	2.747
26	.438	.899	.488	71	.946	.326	2.904
27	.454	.891	.510	72	.951	.309	3.078
28	.470	.883	.532	73	.956	.292	3.271
29	.485	.875	.554	74	.961	.276	3.487
30	.500	.866	.577	75	.966	.259	3.732
31	.515	.857	.601	76	.970	.242	4.011
32	.530	.848	.625	77	.974	.225	4.331
33	.545	.839	.649	78	.978	.208	4.705
34	.559	.829	.675	79	.982	.191	5.145
35	.574	.819	.700	80	.985	.174	5.671
36	.588	.809	.727	81	.988	.156	6.314
37	.602	.799	.754	82	.990	.139	7.115
38	.616	.788	.781	83	.993	.122	8.144
39	.629	.777	.810	84	.995	.105	9.514
40	.643	.766	.839	85	.996	.087	11.43
41	.658	.755	.869	86	.998	.070	14.30
42	.669	.743	.900	87	.999	.052	19.08
43	.682	.731	.933	88	.999	.035	28.64
44	.695	.719	.966	89	1.000	.018	57.29
45	.707	.707	1.000	90	1.000	.000	∞

Appendices

Chapter 1

1-8. Set Notation

A set is a collection of objects - not necessarily material objects - described in such a way that there is no doubt as to whether a particular object does or does not belong to the set.

We use capital letters (A, B, ...) as names of sets. In particular, R is the name of the set of all real numbers.

A set may be described by listing its elements within braces, as

$$A = \{1, 2, 3, 4\},$$

or by using the set-builder notation, as

$$A = \{x: x \text{ is a positive integer and } x < 5\}.$$

(this should be read "A is the set of all x such that x is a positive integer and x is less than 5.")

The Greek letter $\in$(epsilon) is used to indicate that an element belongs to a given set, as

$$2 \in A.$$

(Read this "2 is an element of the set A" or "2 belongs to the set A.")

The intersection of two sets A and B, written $A \cap B$, is the set of all elements that belong to A and also belong to B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The union of two sets A and B, written $A \cup B$, is the set of all elements that belong to A or to B or to both:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

More extensive discussions of sets can be found in the following:

Report of the Commission on Mathematics - Appendices, College Entrance Examination Board, 1959, Chapters 1, 2, 9.

The Growth of Mathematical Ideas, Grades K-12, 24th Yearbook, NCTM, 1959, Chapter 3.

Insights into Modern Mathematics, 23rd Yearbook, NCTM, 1937, Chapter 3.

Elements of Modern Mathematics, K. O. May, Addison-Wesley Publishing Co., Reading, Mass., 1959, Chapter 3.

Fundamentals of Freshman Mathematics, C. B. Allendoerfer and C. O. Oakley, McGraw-Hill Book Co., New York, 1959, Chapter 6.

Introduction to the Theory of Sets, J. Breuer, translated by H. F. Fehr, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1958.

Appendices

Chapter 2

2-11. Mathematical Induction

The primary art of the creative mathematician is to form general hypotheses in the light of a limited number of facts. Secondary, perhaps, but equally essential, is the art of proof. The successful mathematician is one who can make good guesses, by which we mean guesses that he can prove. The best way to show how to guess at a general principle from limited observations is to give examples.

Example 1. Consider the sums of consecutive odd positive integers:

$$\begin{aligned} 1 &= 1 \\ 1 + 3 &= 4 \\ 1 + 3 + 5 &= 9 \\ 1 + 3 + 5 + 7 &= 16 \\ 1 + 3 + 5 + 7 + 9 &= 25 \end{aligned}$$

Notice that in each case the sum is the square of the number of terms.

Conjecture. The sum of the first n odd positive integers is n^2. (This is true; can you show it?)

Example 2. Consider the following inequalities:

$$1 < 100, \quad 2 < 100, \quad 3 < 100, \quad 4 < 100, \quad 5 < 100.$$

Conjecture. All positive integers are less than 100. (False, of course.)

Example 3. Consider the number of complex zeros, repetitions counted, for polynomial functions of various degrees.

zero degree: $x \to a_0,\ a_0 \neq 0$, no zeros

first degree: $x \to a_1 x + a_0,\ a_1 \neq 0$, one zero at $x = \dfrac{-a_0}{a_1}$

second degree: $x \to a_2 x^2 + a_1 x + a_0,\ a_2 \neq 0$, two zeros at

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$$

Conjecture. Every polynomial function of degree n has exactly n complex zeros when repetitions are counted. (True. See Section 2-8.)

Example 4. Observe the operations necessary to compute the zeros from the coefficients in Example 3.

Conjecture. The zeros of a polynomial function of degree n can be given in terms of the coefficients by a formula involving only addition, subtraction, multiplication, division and the extraction of roots. (False. See Section 2-1.)

Example 5. Take any even number except 2 and try to express it as the sum of as few primes as possible:

$4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 5 + 5$, $12 = 5 + 7$, $14 = 7 + 7$, etc.

Conjecture. Every even number except 2 can be expressed as the sum of two primes. (As yet, no one has been able to prove or disprove this conjecture.)

Common to all these examples is the fact that we are trying to assert something about all the members of a sequence of things: the sequence of odd integers, the sequence of positive integers, the sequence of degrees of polynomials functions, the sequence of even numbers greater than 2. The sequential character of the problems naturally leads to the idea of sequential proof. If we know something is true for the first few members of the sequence, can we use that result to prove its truth for the next member of the sequence? Having done that, can we now carry the proof on to one more member? Can we repeat the process again, and again, and again?

Let us try the idea of sequential proof on Example 1. Suppose we know that for the first k odd integers $1, 3, 5, \ldots, 2k-1$,

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2, \qquad (1)$$

can we prove that upon adding the next higher odd number $(2k + 1)$ we obtain the next higher square? From (1) we have at once by adding $2k + 1$ on both sides,

$$[1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2.$$

It is clear that if the conjecture of Example 1 is true at any stage then it is true at the next stage. Since it is true for the first stage, it must be true for the second stage, therefore true for the third stage, hence the fourth, the fifth, and so on forever.

Example 6. In many good toy shops there is a puzzle that consists of three pegs and a set of graduated discs as depicted in Fig. 2-11a. The idea is to transfer the pile of discs from one peg to another under the following rules:

a) Only one disc may be moved at a time.

b) No disc may be placed over a smaller disc.

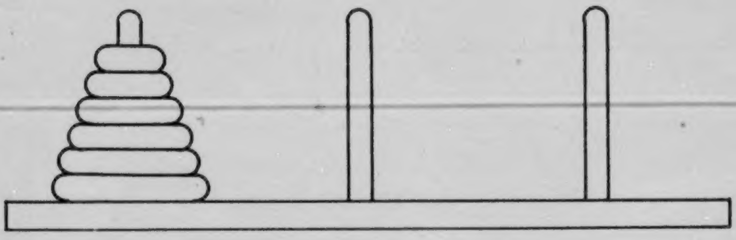

Figure 2-11a.

Two questions arise naturally: Is it possible to execute the task under the stated restrictions? If it is possible, how many moves does it take to complete the transfer of the discs? If it were not for the idea of sequential proof, one might have difficulty in attacking these questions.

As it is, we observe that there is no problem in transferring one disc.

If we have to transfer two discs, we transfer one, leaving a peg for the second disc; we transfer the second disc and cover with the first.

If we have to transfer three discs, we transfer the top two, as above. This leaves a peg for the third disc to which it is then moved, and the first two discs are then transferred to cover the third disc.

The pattern has now emerged. If we know how to transfer k discs, we can transfer k + 1 in the following way: first, we transfer k discs leaving the last disc free to move to a new peg; we move the last disc and then transfer the k discs again to cover the bottom disc. We see then, that it is possible to move any number of graduated discs from one peg to another without violating the rules (a) and (b) since, knowing how to move one disc, we have a rule that tells us how to transfer two, and then how to transfer three, and so on.

To determine the smallest number of moves it takes to transfer a pile of discs, we observe that no disc can be moved unless all the discs above it have been transferred, leaving a free peg to which to move it. Let us designate by m_k the minimum number of moves needed to transfer k discs. To move the (k + 1)th disc we first need m_k moves to transfer the discs above it to another peg. After that, we can transfer the (k + 1)th disc to the free peg. To move the (k + 2)th disc (or to conclude the game if the (k + 1)th disc is last), we must now cover the (k + 1)th disc with the preceding k discs, and this transfer of the k discs cannot be accomplished in less than m_k moves. We see then that the minimum number of moves for k + 1 discs is

$$m_{k+1} = 2m_k + 1.$$

This is a recursive expression for the minimum number of moves, that is, if the minimum is known for a certain number of discs, we can calculate the minimum for one more disc. In this way we have defined the minimum number of moves sequentially: by adding one disc we increase the necessary number of moves to one more than twice the preceding number. It takes one move to move one disc, therefore, it takes three moves to move two discs, etc.

Let us make a little table, as follows:

k	1	2	3	4	5	6	7
m_k	1	3	7	15	31	63	127

k = number of discs
m_k= minimum number of moves

Upon adding a disc we roughly double the number of moves. This leads us to compare the number of moves with the powers of two: 1, 2, 4, 8, 16, 32, 64, 128, ..., and we guess that $m_n = 2^n - 1$. If this is true for some value k, we can easily see that it must be true for the next, for we have

$$\begin{aligned} m_{k+1} &= 2m_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1, \end{aligned}$$

and this is the value of $2^n - 1$ for $n = k + 1$. We know that the formula for m_k is valid when $k = 1$, but now we can prove in sequence that it is true for 2, 3, 4, and so on.*

The principle of sequential proof, stated explicitly, is this:

First Principle of Mathematical Induction: Let $A_1, A_2, A_3, \ldots$ be a sequence of assertions, and let H be the hypothesis that all of these are true. The hypothesis H will be accepted as proved if:

1) There is a general proof to show that if any assertion A_k is true, then the next assertion A_{k+1} is true.

2) There is a special proof to show that A_1 is true.

If there are only a finite number of assertions in the sequence, say ten, then we need only carry out the chain of ten proofs explicitly to have a complete proof. If the assertions continue in sequence endlessly, as in Example 1, then we cannot possibly verify directly every link in the chain of proof. It is just for this reason--in effect that we can handle an infinite chain of proofs without specifically examining every link--that the concept of sequential proof becomes so valuable. It is, in fact at the heart of the logical development of mathematics.

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*According to persistent rumor, there is a puzzle of this kind in a most holy Buddhist monastery hidden deep in the Himalayas. The game consists of sixty-four discs of pure beaten gold and the pegs are diamond needles. The story relates that the game of transferring the discs has been played by the monks since the beginning of time, day and night, and has yet to be concluded. It has also been said that when the sixty-four discs are completely transferred, the world will come to an end. [The physicists say the earth is about four billion years old, give or take a billion or two. Assuming that the monks move one disc every second and play in the minimum number of moves, is there any cause for panic?]

Through an unfortunate association of concepts this method of sequential proof has been named "mathematical induction." Induction, in its common English sense, is the guessing of general propositions from a number of observed facts. This is the way one arrives at assertions to prove. "Mathematical induction" is actually a method of deduction or proof and not a procedure of guessing, although to use it we ordinarily must have some guess to test. This usage has been in the language for a long time and we would gain nothing by changing it now. Let us keep it then, and remember that mathematical usage is special and often does not resemble in any respect the usage of common English.

In Example 1, above, the assertion A_n is

$$1 + 3 + 5 + \dots + (2n - 1) = n^2.$$

We proved, first, that if A_k is true, that is, if the sum of the first k odd numbers is k^2, then A_{k+1} is true, so that the sum of the first $k + 1$ odd numbers is $(k + 1)^2$. Secondly, we observed that A_1 is true: $1 = 1^2$. These two steps complete the proof.

Mathematical induction is a method of proving a hypothesis about a list or sequence of assertions. Unfortunately, it doesn't tell us how to make the hypothesis in the first place. In the example just considered, it was easy to guess, from a few specific instances, that the sum of the first n odd numbers is n^2, but the next problem may not be so obvious.

Example 7. Consider the sum of the squares of the first n positive integers,

$$1^2 + 2^2 + 3^2 + \dots + n^2.$$

We find that when $n = 1$, the sum is 1, when $n = 2$, the sum is 5, when $n = 3$, the sum is 14, and so on. Let us make a table of the first few values:

n	1	2	3	4	5	6	7	8
Sum	1	5	14	30	55	91	140	204

Though some mathematicians might immediately be able to see a formula that will give us the sum, most of us would have to admit that the situation is obscure. We must look around for some trick to help us discover the pattern that is surely there, and what we do will therefore be a personal, individual matter. It is a mistake to think that only one approach is possible.

Sometimes experience is a useful guide. Do we know the solutions to any similar problems? Well, we have here the sum of a sequence, and Example 1 also dealt with the sum of a sequence: the sum of the first n odd numbers is n^2. How about the sum of the first n integers themselves (not their squares)? What is

$$1 + 2 + 3 + \dots + n?$$

This seems to be a related problem, and we can solve it with ease. The terms form an arithmetic progression, in which the first term is 1 and the common difference is also 1; the sum, by the usual formula, is therefore $\frac{n}{2}(n + 1) = \frac{1}{2}n^2 + \frac{1}{2}n$. So we have

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n^2 + \frac{1}{2}n.$$

Is there any pattern here that might help with our present problem?

These two formulas have one common feature: both are quadratic polynomials in n. Might not the formula we want here also be a polynomial? It seems unlikely that a quadratic polynomial could do the job in this more complicated problem, but how about one of higher degree? Let's try a cubic: assume that there is a formula

$$1^2 + 2^2 + \dots + n^2 = an^3 + bn^2 + cn + d,$$

where a, b, c, and d are numbers yet to be determined. Substituting $n = 1, 2, 3,$ and 4 successively in this formula, we get

$$1^2 = a + b + c + d$$

$$1^2 + 2^2 = 8a + 4b + 2c + d$$

$$1^2 + 2^2 + 3^2 = 27a + 9b + 3c + d$$

$$1^2 + 2^2 + 3^2 + 4^2 = 64a + 16b + 4c + d$$

Solving, we find

$$a = \frac{1}{3},\ b = \frac{1}{2},\ c = \frac{1}{6},\ d = 0.$$

We therefore conjecture that

$$\begin{aligned} 1^2 + 2^2 + \dots + n^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\ &= \frac{1}{6}n(n + 1)(2n + 1). \end{aligned}$$

This, then, is our assertion A_n; now let us prove it.

We have A_k:

$$1^2 + 2^2 + \dots + k^2 = \frac{1}{6}k\,(k + 1)(2k + 1).$$

Add $(k + 1)^2$ to both sides, factor, and simplify:

$$\begin{aligned} 1^2 + 2^2 + \dots + k^2 + (k + 1)^2 &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1)[\frac{1}{6}k(2k+1) + (k+1)] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3), \end{aligned}$$

and this last equation is just A_{k+1}, which is therefore true if A_k is. Moreover, A_1, which states

$$1^2 = \frac{1}{6}(1)(2)(3),$$

is true, and A_n is therefore true for each positive integer n.

There is another extremely useful formulation of the principle of mathematical induction. This form involves the assumption in the sequential step that every assertion up to a certain point is true, rather than just the one assertion immediately preceding. Specifically, we have this:

Second Principle of Mathematical Induction. Again, let $A_1, A_2, A_3, \dots$ be a sequence of assertions and let H be the hypothesis that all of these are true. The hypothesis H will be accepted as proved if

1) There is a general proof to show that, if every preceding assertion $A_1, A_2, \ldots, A_k$ is true, then the next assertion A_{k+1} is true.

2) There is a special proof to show that A_1 is true.

It is not hard to show that either principle of mathematical induction can be derived from the other, but we shall not do so here.

The value of this form of the principle is that it permits the treatment of many problems which would be quite difficult to handle on the basis of the first principle. Such problems usually present a more complicated appearance than the ones that yield directly to an attack by the first principle.

Example 8. Every set S of natural numbers (whether finite or infinite) contains a least element.*

Proof. The induction is based on the fact that S contains some natural number. The assertion A_k is that if k S then S contains a least element.

Initial Step. The assertion A_1 is that if S contains 1 then it contains a least number. This is certainly true since 1 is the smallest natural number and so is smaller than any other member of S.

Sequential Step: We assume the theorem is true for all natural numbers up to and including k. Now let S be a set containing $k + 1$. There are two possibilities:

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* This example is valuable because it can be used as a third principle of mathematical induction, although not an obvious one to be sure. An amusing example of a "proof" by this principle is given in the American Math. Monthly, Vol. 52(1945) by E. F. Beckenbach.

Theorem: Every natural number is interesting.

Argument: Consider the set S of all uninteresting natural numbers. This set contains a least element. What an interesting number, the smallest in the set of uninteresting numbers! So S contains an interesting number after all. Contradiction.

The trouble with this "proof", of course, is that we have no definition of "interesting"; one man's interest is another man's boredom.

a) S contains a natural number p less than k + 1. In that case p is less than or equal to k. It follows that S contains a least element.

b) S contains no natural number less than k + 1. In that case k + 1 is least.

One of the important uses of mathematical induction is in definition by recursion; that is, we define a sequence of things in the following manner: a definition is given for the initial object of the sequence and a rule is supplied so that if any term is known the rule provides a definition for the succeeding one. For example, we could have defined a^n $(a \neq 0)$ recursively in the following way:

Initial Step, $a^0 = 1$.

Sequential Step: $a^{n+1} = a \cdot a^n \quad n = (0, 1, 2, 3, \ldots)$

Here is another useful definition by recursion:

Let $n!$ denote the product of the first n positive integers. We can define $n!$ recursively as follows:

Initial Step. $1! = 1$.

Sequential Step. $(n + 1)! = (n + 1)(n!)$.

Such definitions are convenient in proofs by mathematical induction. Here is an example involving the two definitions we have just given.

Example 9. For all positive integral values $n, 2^{n-1} \leq n!$ The proof by mathematical induction is direct. We have

Initial Step. $2^0 = 1 \leq 1! = 1$.

Sequential Step. Assuming that the assertion is true at the k-th step, we seek to prove it for the (k + 1)th step. By definition we have

$$(k + 1)! = (k + 1)(k!).$$

From the hypothesis, $k! \geq 2^{k-1}$, and consequently,

$$(k + 1)! = (k+1)(k!) \geq (k+1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k$$

since $k \geq 1$ (k is a positive integer). We conclude that $(k + 1)! \geq 2^k$.

The proof is complete.

Before we conclude these remarks on mathematical induction, a word of caution. For a complete proof by mathematical induction it is important to show the truth of both the initial step and the sequential step of the induction principle being used. There are many examples of mathematical induction gone haywire because one of these steps fails. Here are two examples:

Example 10.

Assertion. All natural numbers are even.

Argument. For the proof we utilize the second principle of mathematical induction and take for A_k the assertion that all natural numbers less than or equal to k are even. Now consider the natural number $k + 1$. Let i be any natural number with $i \leq k$. The number j such that $i + j = k + 1$ can easily be shown to be a natural number with $j \leq k$. But if $i \leq k$ and $j \leq k$, then both i and j are even, and hence $k + 1 = i + j$, the sum of two even numbers, and must itself be even!

Example 11.

Assertion. Every girl has blue eyes.

Argument. Let us begin with one girl, my girl friend, who happens to have blue eyes. Now let us assume that in every set of k girls that includes my girl, all have blue eyes. Consider any set of $k + 1$ girls. If any girl is removed from this set we have k girls left, and hence all the girls left have blue eyes. It might be supposed that the girl removed from the set had brown eyes. Yet it is easy to see that this cannot be true. Leave the girl supposed to have brown eyes in the set and remove some other girl. Since k girls remain, they must all have eyes of the same color, blue eyes by assumption. It follows that all $k + 1$ girls have blue eyes, and the proof is complete.

Find the holes in the two arguments.

Exercises 2-11

1. Prove by mathematical induction:

$$1 + 2 + 3 + \dots + n = \tfrac{1}{2}n(n + 1).$$

2. a) By mathematical induction prove the familiar result giving the sum of an arithmetic progression to n terms:

$$a + (a+d) + (a+2d) + \dots + (a + (n-1)d) = \frac{n}{2}[2a + (n-1)d].$$

b) Do the same for a geometric progression:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

Prove the following four exercises by mathematical induction for all positive integers n:

3. $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$.

4. $2n \leq 2^n$.

5. If $p > -1$ then for every positive integer n,
   $(1 + p)^n \geq 1 + np$.

6. $1 + 2\cdot 2 + 3\cdot 2^2 + \dots + n\cdot 2^{n-1} = 1 + (n-1)2^n$.

7. Prove by the second principle of mathematical induction:
   a) For all natural numbers n, the number $n + 1$ either is a prime or can be factored into primes.
   b) For each natural number $n > 1$, let U_n be a real number with the property that for at least one pair of natural numbers p, q with $p + q = n$,
   $$U_n = U_p + U_q.$$
   When $n = 1$ we define $U_1 = a$ where a is some given real number. Prove that $U_n = na$ for all n.
   c) Attempt to prove (a) and (b) from the first principle to see what difficulties arise.

In the next three problems, first discover a formula for the sum and then prove, by mathematical induction, that your formula is correct.

8. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)}$.

9. $1^3 + 2^3 + 3^3 \dots + n^3$. (Hint: Compare the sums you get here with Example 1 in the text, or alternatively, assume that the required result is a polynomial of degree 4.)

10. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)$. (Hint: Compare this one with Example 8.)

11. In geometry it has been proved of any 3 points, P_1, P_2, P_3, that $m(P_1P_2) + m(P_2P_3) \geq m(P_1P_3)$ (triangle inequality), where $m(P_iP_j)$ denotes the distance between points P_i and P_j. Let $P_1, P_2, \ldots, P_n$ denote any n points in the plane, $n \geq 3$.
Prove that

$$m(P_1P_2) + m(P_2P_3) + m(P_3P_4) + \ldots + m(P_{n-1}P_n) \geq m(P_1P_n).$$

12. Prove that for all positive integers n,

$$(1 + \tfrac{3}{1})(1 + \tfrac{5}{4})(1 + \tfrac{7}{9}) \ldots (1 + \tfrac{2n+1}{n^2}) = (n + 1)^2.$$

*13. Prove that $n(n^2 + 5)$ is divisible by 6 for all integral n.

*14. A band of pirates is sitting in a circle dividing up their loot. One man is their leader. According to their code, each man in the circle must get the arithmetic average of the amounts received by the two men on his right and left. This rule does not apply to the leader. In what proportions may they divide the loot?

*15. Consider the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{p_n}{q_n}, \ldots$$

where each fraction is obtained from the preceding by the rule

$$p_n = p_{n-1} + 2q_{n-1}$$
$$q_n = p_{n-1} + q_{n-1}$$

Show that for n sufficiently large, the difference between p_n/q_n and $\sqrt{2}$ can be made as small as desired. Show also that the approximation to $\sqrt{2}$ is improved at each successive stage of the sequence and that the error alternates in sign. Prove also that p_n and q_n are relatively prime: that is, the fraction p_n/q_n is in lowest terms.

**16. Let p be any polynomial function of degree m. Let $q(n)$ denote the sum

$$q(n) = p(1) + p(2) + p(3) + \ldots + p(n). \qquad (1)$$

Prove that there is a polynomial q of degree $m+1$ satisfying (1).

**17. Let the function f be defined recursively as follows:

Initial Step. $f(1) = 3$

Recursive Step. $f(n + 1) = 3^{f(n)}$

In particular, we have

$$f(3) = 3^{3^3} = 3^{27}, \text{ etc.}$$

Similarly, let $g(n)$ be defined thus:

Initial Step. $g(1) = 9$

Recursive Step. $g(n+1) = 9^{g(n)}$.

Find the minimum value m for each n such that

$$f(m) \geq g(n).$$

Answer: $m = n + 1$.

**18. Prove that, for all natural numbers n,

$$\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

is an integer. (Hint: Try to express $x^n - y^n$ in terms of $x^{n-1} - y^{n-1}$, $x^{n-2} - y^{n-2}$, etc.)

Appendices

Chapter 2

2-12. Significance of Polynomials

The importance of polynomials in applications to engineering and the natural sciences, as well as in the body of mathematics itself, is not an accident. The utility of polynomials is based largely on mathematical properties that, for all practical purposes, permit the replacement of much more complicated functions by polynomial functions in a host of situations. We shall enumerate some of these properties:

a) Polynomial functions are among the simplest functions to manipulate formally. The sum, product, and composite of polynomial functions, the determination of slope and area, and the location of zeros and maxima and minima are all within the reach of elementary methods.

b) Polynomial functions are among the simplest functions to evaluate. It is quite easy to find the value of $f(x)$, given

$$f: x \longrightarrow a_0 + a_1x + a_2x^2 + \dots + a_nx^n \qquad (1)$$

with a specific set of coefficients $a_0, a_1, \dots, a_n$ and a specific number for x. Nothing more than multiplication and addition is involved, and the computation can be shortened by using the method of synthetic substitution.

The foregoing two properties of polynomial functions are those that make them valuable as replacements for more complicated functions.

c) Frequently an experimental scientist makes a series of measurements, plots them as points, and then tries to find a reasonably simple continuous curve that will pass through these points. The graph of a polynomial function can always be used for this purpose, and because it has no sharp changes of direction, and only a limited number of ups and downs, it is in many ways the best curve for the purpose.

Thus, for the purpose of fitting a continuous graph to a finite number of points, we would prefer to work with polynomials and we need not look beyond the polynomials, as we shall prove. We can state the problem formally as follows:

Given n distinct numbers $x_1, x_2, \ldots, x_n$ and corresponding values $y_1, y_2, \ldots, y_n$ that a function is supposed to assume, it is possible to find a polynomial function of degree at most $n - 1$ whose graph contains the n points (x_i, y_i), $i = 1, \ldots, n$. You have already done this for $n = 2$: you found a linear or constant function whose graph contained two given points (x_1, y_1), (x_2, y_2), $x_2 \neq x_1$. If y_2 is also different from y_1, the result is a linear function; if $y_2 = y_1$, it is a constant function.

One way of doing this is to assume a polynomial of the stated form,

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1},$$

and write the n equations

$$f(x_i) = y_i, \quad i = 1, 2, \ldots, n.$$

This gives n linear equations in the n unknowns $a_0, a_1, \ldots, a_{n-1}$, and in these circumstances such a system will always have a solution.

Example 1. Suppose that we want the graph of a function to pass through the points $(-2,2)$, $(1,3)$, $(2,-1)$, and $(4,1)$. We know that there is a polynomial graph of degree no greater than 3 which goes through these points. Assume, therefore,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

Then, if the graph of f is to go through the given points, we must have $f(-2) = 2$, $f(1) = 3$, $f(2) = -1$, and $f(4) = 1$; that is,

$$\begin{aligned}
a_0 - 2a_1 + 4a_2 - 8a_3 &= 2,\\
a_0 + a_1 + a_2 + a_3 &= 3,\\
a_0 + 2a_1 + 4a_2 + 8a_3 &= -1,\\
a_0 + 4a_1 + 16a_2 + 64a_3 &= 1.
\end{aligned}$$

Solving these, we find $a_0 = 20/3$, $a_1 = -31/12$, $a_2 = -37/24$, and $a_3 = 11/24$. Hence

$$f(x) = \frac{1}{24}(160 - 62x - 37x^2 + 11x^3).$$

The labor of solving systems of linear equations such as these can be rather discouraging, especially if there are many equations. For this reason, various methods have been worked out for organizing and reducing the labor involved. One of the most important of these methods, called the Lagrange Interpolation Formula, is based on the following simple line of reasoning. We can easily write down a formula for a polynomial of degree $n-1$ that is zero at $n-1$ of the given x's.

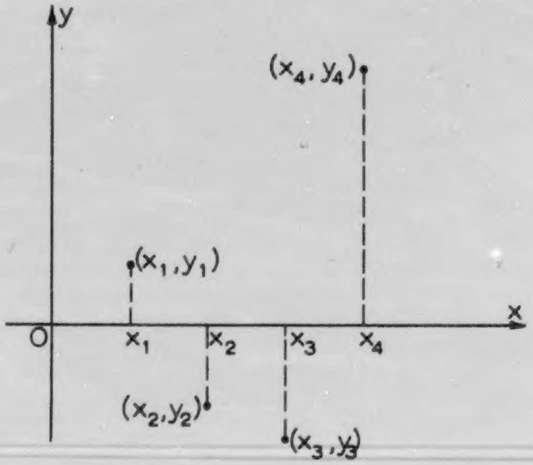


Figure 2-12a

A set of values to be taken on by a polynomial function.

Suppose, for instance, that we have four points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) as in Figure 2-12a. The polynomial

$$g_1(x) = C_1(x - x_2)(x - x_3)(x - x_4) \qquad (1)$$

has zeros at x_2, x_3, and x_4. By proper choice of C_1, we can make $g_1(x_1) = y_1$. Let us do so! Take C_1 such that

$$y_1 = g_1(x_1) = C_1(x_1 - x_2)(x_1 - x_3)(x_1 - x_4),$$

that is, take

$$C_1 = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}. \qquad (2)$$

If we substitute C_1 from (2) into (1), we get

$$g_1(x) = y_1 \cdot \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} \tag{3}$$

If $y_1 \neq 0$, Equation (3) defines a polynomial of degree 3 that has the value y_1 at x_1 and is zero at x_2, x_3 and x_4. Similarly, one finds that

$$g_2(x) = y_2 \cdot \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}, \tag{4}$$

$$g_3(x) = y_3 \cdot \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)}, \tag{5}$$

and

$$g_4(x) = y_4 \cdot \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}, \tag{6}$$

are also polynomials, each having the property that it is zero at three of the four given values of x, and is the appropriate y at the fourth x. This is shown in the table below.

The Lagrange Interpolation Formula Illustrated.

Values of x	x_1	x_2	x_3	x_4
Corresponding y	y_1	y_2	y_3	y_4
Value of $g_1(x)$	y_1	0	0	0
Value of $g_2(x)$	0	y_2	0	0
Value of $g_3(x)$	0	0	y_3	0
Value of $g_4(x)$	0	0	0	y_4

If we form the sum

$$g(x) = g_1(x) + g_2(x) + g_3(x) + g_4(x), \tag{7}$$

then it is clear from the table that

$$g(x_1) = y_1 + 0 + 0 + 0 = y_1,$$

$$g(x_2) = 0 + y_2 + 0 + 0 = y_2,$$

$$g(x_3) = 0 + 0 + y_3 + 0 = y_3,$$

$$g(x_4) = 0 + 0 + 0 + y_4 = y_4.$$

From Equations (3), (4), (5), and (6) it is also clear that g is a polynomial in x whose degree is at most 3. Hence Equation (7) tells us how to find a polynomial of degree ≤ 3, whose graph contains the given points.

Example 2. Find a polynomial of degree at most 3 whose graph contains the points $(-1,2)$, $(0,0)$, $(2, -1)$, and $(4,2)$.

Solution. We find that

$$g_1(x) = 2\left\{\frac{(x - 0)(x - 2)(x - 4)}{(-1-0)(-1-2)(-1-4)}\right\} = \frac{2x(x - 2)(x - 4)}{- 15},$$

$$g_2(x) = 0,$$

$$g_3(x) = -1 \left\{\frac{(x + 1)(x - 0)(x - 4)}{(2 + 1)(2 - 0)(2 - 4)}\right\} = \frac{(x + 1)(x)(x - 4)}{12},$$

$$g_4(x) = 2 \left\{\frac{(x + 1)(x - 0)(x - 2)}{(4 + 1)(4 - 0)(4 - 2)}\right\} = \frac{(x + 1)(x)(x - 2)}{20},$$

and

$$g(x) = - \frac{2}{15}x(x-2)(x-4) + \frac{1}{12}(x+1)x(x-4) + \frac{1}{20}(x+1)(x)(x-2)$$

$$= \frac{1}{2}(x^2 - 3x).$$

Remark. The right-hand sides of Equations (3), (4), (5), and (6) have the following structure:

$$g_i(x) = y_i \frac{N_i(x)}{D_i} \; ; \quad i = 1, 2, 3, 4.$$

The numerator of the fraction is the product of all but one of the factors

$$(x - x_1),\ (x - x_2),\ (x - x_3), \text{ and } (x - x_4),$$

and the missing factor is $(x - x_i)$. The denominator is the value of the numerator at $x = x_i$;

$$D_i = N_i(x_i).$$

This same structure would still hold if we had more (or fewer) points given.

d) Instead of a finite set of points to which a simple continuous function is to be fitted, a mathematician is sometimes confronted with a continuous but very complicated function that he would like to approximate by a simpler function. Fortunately, there is an extremely powerful theorem of higher mathematics that enlarges the breadth of application of polynomials to this situation. In a sense this theorem permits the "fitting" of a polynomial graph to any continuous graph. In other words, any continuous function whatever can be approximated by a polynomial function over a finite interval of its domain, with preassigned accuracy. More specifically, if the function $x \longrightarrow f(x)$ is continuous over $a \leq x \leq b$, and c is any positive number, there exists a polynomial function g such that

$$|f(x) - g(x)| < c \text{ for all } x \text{ in } a \leq x \leq b.$$

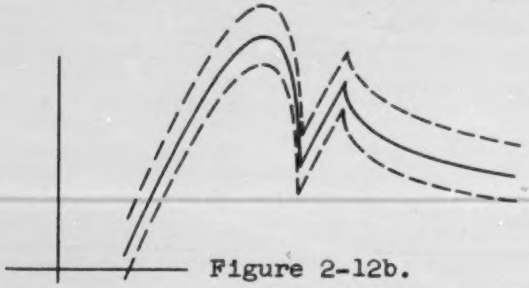


Figure 2-12b.

A strip between $f(x)-c$ and $f(x)+c$.

This is known as the Weierstrass Approximation Theorem. The geometric interpretation of the theorem is indicated in Figure 2-12b. The graph of f is a continuous curve, but it may have sharp corners or even infinitely many maxima and minima between $x = a$ and $x = b$. No polynomial graph behaves like that. But suppose that we introduce a strip, centered on the graph of f, extending between the graphs of the functions

$$x \longrightarrow f(x) - c$$

and

$$x \longrightarrow f(x) + c,$$

where c is any preassigned positive number, however small. Then the theorem guarantees that there is a polynomial function

$$g: x \longrightarrow g(x),$$

whose graph on $a \leq x \leq b$ lies entirely inside this strip. This

[sec. 2-12]

is the precise meaning of the statement: "Any continuous function whatever can be approximated by a polynomial function over a finite interval of its domain, with preassigned accuracy."

Remark. If you want to read more about how polynomials are used to approximate more complicated functions, see the book Approximations for Computers, by Cecil Hastings, Jr., Princeton University Press, 1955. The author also publishes a newsletter called "All the Fit That's News to Print" and supplies people in the computing business with simple functions that may be used to replace more complicated ones.

Exercises 2-12.

1. Carry out the computations in Example 1, above.
2. Simplify the expression for $g(x)$ in Example 2, above.
3. Find a polynomial function of degree less than or equal to 2 whose graph contains the points $(-1,2)$, $(0,-1)$, $(2,3)$.
4. Find a polynomial function whose graph contains the points $(0,1)$, $(1,0)$, $(2,9)$, $(3,34)$ and $(-1,6)$.

3-12. Area Under A Polynomial Graph

In most of the preceding sections of this chapter our attention was focused on the problem of finding slopes of graphs of polynomial functions. As we were able to observe, the solution of this problem had important consequences. By developing a technique for finding slopes of graphs, for example, we were able to solve maximum and minimum problems.

In the present section we turn to another problem, that of calculating areas bounded by graphs of polynomial functions. The solution of this problem has important consequences also. It leads directly to the general concept of integration, which is an indispensable tool for advanced work in much of mathematics and science. The extended study of this key concept must, however, await further developments in your mathematical education.

For simplicity we shall concentrate our attention on finding areas of certain special kinds of regions. These regions will be located in the first quadrant and bounded by the graph of a polynomial function f, the x-axis, the y-axis, and a second vertical line, as in Figure 3-12a. (It is not too hard to see that any region bounded by graphs of polynomial functions can be split up into regions of this general shape.) Note that we do not specify the value of the coordinate x at which the second vertical line cuts the x-axis. This will allow us to find general formulas rather than particular numbers, and thus will lead to greater

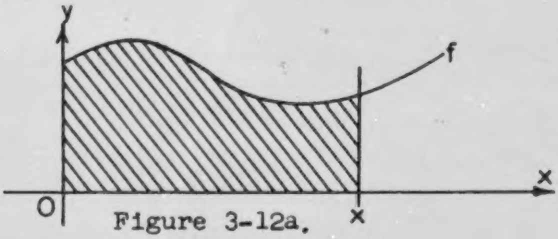


Figure 3-12a.
Area under a graph

understanding. In what follows we shall denote the desired area by $A(x)$.

Usually the first step a mathematician will take in attacking a new problem is to investigate a few special cases of the problem. He generally finds this investigation very helpful in getting the "feel" of the problem and in setting his mind working toward a general solution. In this spirit we begin with the simplest of the polynomial functions and examine the area under the graph of a constant function $x \longrightarrow c$, where c is a fixed positive number. This case is very easy to handle. In fact, since we know that the area of a rectangle is equal to the product of its base and its height, we see that the desired area is $A(x) = cx$. (See Figure 3-12b.)

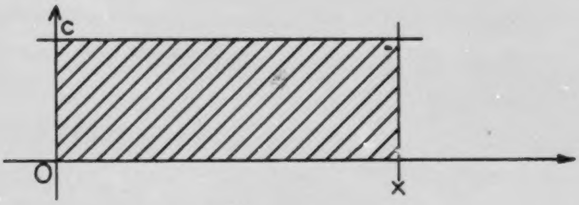


Figure 3-12b.

Area under $f: x \longrightarrow c$.

Note that the "area function" $x \longrightarrow cx$ is a linear function whose slope function is $f: x \longrightarrow c$.

In order of complexity, the next case we should examine is that of a linear function $f: x \longrightarrow mx + b$. This is illustrated in Figure 3-12c.

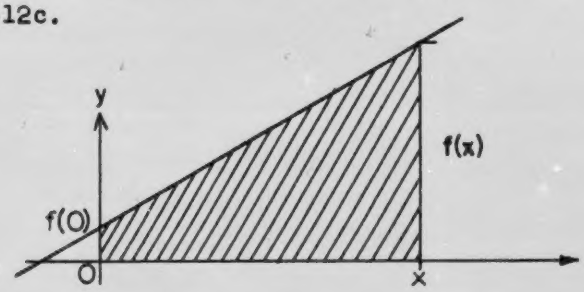


Figure 3-12c.

Area under $f: x \longrightarrow mx + b$.

This case, too is easy to handle. Since the area of a trapezoid is one half the sum of the bases times the height (and noting that the "height" of our trapezoid is the length of the horizontal segment from 0 to x) we have

$$\begin{aligned} A = A(x) &= \frac{f(0) + f(x)}{2} \cdot x \\ &= \frac{(m \cdot 0 + b) + (mx + b)}{2} \cdot x \\ &= \frac{mx + 2b}{2} \cdot x \\ &= (mx^2/2) + bx. \end{aligned}$$

Note that the "area function" $x \longrightarrow (mx^2/2) + bx$ is a quadratic function whose slope function is $f: x \longrightarrow mx + b$.

After the constant functions and the linear functions, the next simplest polynomial functions are the quadratic functions. Even though these functions seem to be but a step removed from the linear functions, we shall see that they introduce an entirely new order of complexity. The reason for this is that the graphs of quadratic functions are curves, and we have no formulas for calculating areas of regions bounded by curves (except, of course, when the curves are circular). Hence it will be wise to move more slowly, and first study a very special case--say the function $f: x \longrightarrow x^2$. (See Figure 3-12d.)

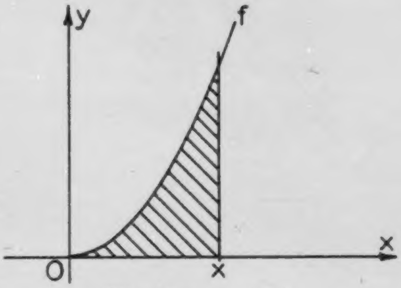


Figure 3-12d.

Area under $f: x \longrightarrow x^2$

If it were possible to cut the region up into a finite number of rectangular or triangular parts we could add up the areas of the parts to obtain the total area. There is, however, no way to do this. The best we can do is to approximate the area. We can cover the region with rectangles and obtain in the sum of their areas a value that is somewhat larger than the one we seek. On the other hand, we can pack rectangles into the region without overlapping, and obtain in the sum of their areas a value that is somewhat too small. In this way we may at least hope to arrive at an approximate value that we might be able to use in constructing our area function.

A systematic way to approximate the area by means of rectangles is illustrated in Figure 3-12e.

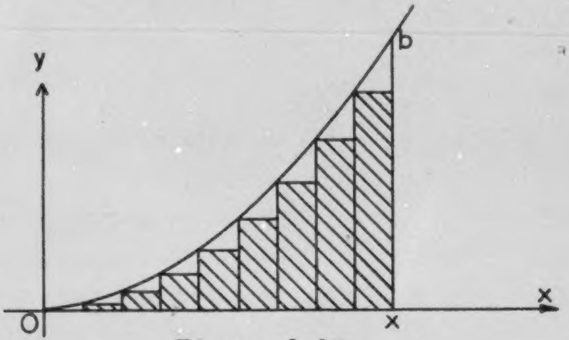


Figure 3-12e.

Area approximated by interior rectangles.

The procedure we shall follow is as follows: First we shall split the line segment from 0 to x into n equal parts or subintervals, where n is some unspecified positive integer. Each of these subintervals will be the base of a rectangle, the largest rectangle that can be drawn under the curve with this subinterval as a base. (See Figure 3-12f.)

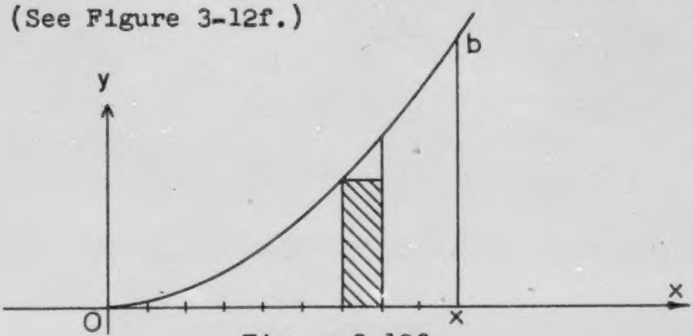


Figure 3-12f.

Area of subinterval approximated by interior rectangle.

We shall then calculate the sum of the areas of all these rectangles to obtain an approximation to the area between the graph and the x-axis.

If we split the line segment from 0 to x into n equal parts, the length of each part will be x/n and the end points of the parts will be the numbers

$$0,\ \frac{x}{n},\ 2(\frac{x}{n}),\ 3(\frac{x}{n}),\ \ldots\ ,\ (n-1)(\frac{x}{n}),\ n(\frac{x}{n}).$$

The graph has a height above each of these end points. The heights corresponding to these end points are

$$f(0),\ f(x/n),\ f(2x/n),\ \ldots,\ f(nx/n),\ \text{respectively,}$$

or

$$0,\ x^2/n^2,\ 4x^2/n^2,\ \ldots\ ,\ n^2x^2/n^2,\ \text{respectively.}$$

With the first subinterval as a base no rectangle can be drawn, because the curve touches this subinterval at the origin.

With the second subinterval as a base the height of the largest rectangle that can be drawn under the curve is $f(x/n) = x^2/n^2$. Since the base is x/n, the area of this rectangle is $(x^2/n^2)(x/n) = x^3/n^3$.

With the third subinterval as a base the height of the largest rectangle that can be drawn under the curve is $f(2x/n) = 4x^2/n^2$. Since the base is x/n, the area of this rectangle is $(4x^2/n^2)(x/n) = 4x^3/n^3$.

Continuing in this fashion we see that the sum of the areas of all these rectangles must be

$$(x^2/n^2)(x/n) + (4x^2/n^2)(x/n) + (9x^2/n^2)(x/n) + \ldots$$
$$+ [(n-1)^2\ x^2/n^2]\ (x/n).$$

If we factor x^3/n^3 from each term, this may be written

$$\frac{x^3}{n^3}\ [\ 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2]$$

and, as we saw in Example 7 of Section 2-11 of appendices, the sum of squares in this is

$$\frac{1}{6}(n-1)(n)(2n-1) = \frac{1}{6}(2n^3 - 3n^2 + n).$$

Hence the sum of the areas of all these rectangles is

$$\frac{x^3}{n^3} \cdot \frac{1}{6}(2n^3 - 3n^2 + n)$$

Since the rectangles we have been dealing with are all drawn below the curve, it is clear that this approximation is too small. (See Figure 3-12e.) It is desirable, therefore, to obtain another approximation by finding the combined area of rectangles that extend above the curve. This will give us too large an approximation, and the desired area will lie between these two approximations. By calculating the difference between the two approximations, therefore, we shall get an idea of how far off these approximations really are. To obtain a useful approximation that is too large, we find the sum of rectangles as illustrated in Figure 3-12g.

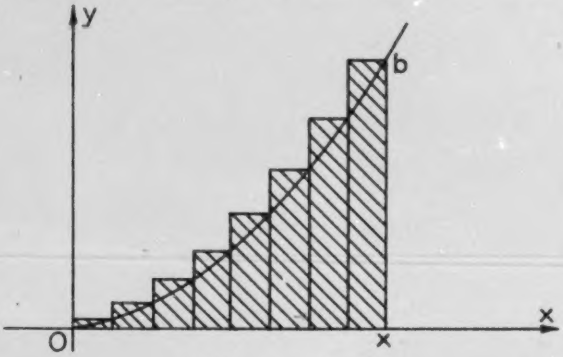


Figure 3-12g.

Area approximated by exterior rectangles.

This sum is obtained in almost exactly the same way as the one we just obtained. This over-estimate is

$$\begin{aligned} &(x^3/n^3)\ \left[1^2 + 2^2 + 3^2 + \dots + (n-1)^2 + n^2\right] \\ = &(x^3/n^3)\ \left[(1/6)\ (2n^3 - 3n^2 + n) + n^2\right] \\ = &(x^3/n^3)(1/6)(2n^3 + 3n^2 + n). \end{aligned}$$

Now the difference between the two estimates is only the single term $(x^3/n^3)n^2 = x^3/n$, which is small if n is large compared to x. Hence either estimate will be quite good when n is very large.

To determine what the desired area $A(x)$ must be, let us express the under-estimate as

$$\frac{x^3}{6n^3}(2n^3 - 3n^2 + n)$$

and multiply through to obtain

$$\frac{x^3}{3} - \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

Similarly, the over-estimate is

$$\frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

Given the fixed number x, if n is very large compared with x the last two terms of each expression must be very small. The only value that the area $A(x)$ can have, therefore, is $x^3/3$. Note, finally, that the "area function" $x \longrightarrow x^3/3$ is a function whose slope function is $f: x \longrightarrow x^2$.

3-13. Slopes of Area Functions

In Section 3-12 we examined the problem of finding the area under the graph of a polynomial function. For simplicity we concentrated our attention on finding the area of a region located in the first quadrant and bounded by the graph of a polynomial function f, the x-axis, the y-axis and a second vertical line, as in Figure 3-13a.

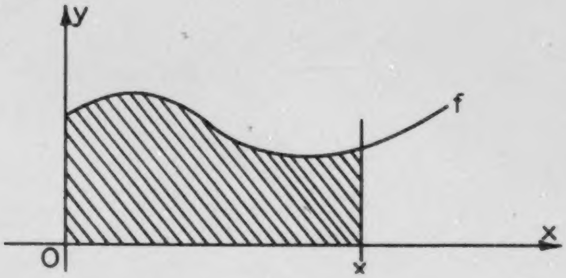


Figure 3-13a.

Area under a graph

In order to find general formulas rather than particular numbers, we did not specify the value of the coordinate x at which the second vertical line cuts the x-axis. Calling the indicated area $A(x)$, we obtained a function $x \longrightarrow A(x)$, which we called the "area function." In what follows we shall designate this area function by the letter A,

$$A: x \longrightarrow A(x).$$

The results we obtained in Section 3-12 can be tabulated as follows:

Function f	Area function A associated with f	Slope function A' of area function
$x \longrightarrow c$	$x \longrightarrow cx$	$x \longrightarrow c$
$x \longrightarrow mx + b$	$x \longrightarrow \frac{mx^2}{2} + bx$	$x \longrightarrow mx + b$
$x \longrightarrow x^2$	$x \longrightarrow \frac{x^3}{3}$	$x \longrightarrow x^2$

It is impossible to miss the similarity between the first and third columns of this table. Since these two columns are identical except for heading, we are practically compelled to suspect that there must be some relationship between a function f and the slope function A' of its associated function A. With evidence of this sort to support us, we boldly conjecture:

If A is the area function associated with a function f, then $A' = f$.

In Figure 3-13c, we have plotted the graph of the area function

$$A: x \longrightarrow A(x)$$

associated with the function

$$f: x \longrightarrow f(x)$$

shown in Figure 3-13b.

We wish to prove that at $P(h,A(h))$, the slope of the tangent to this graph is $f(h)$. We shall confine our attention to points to the right of P.

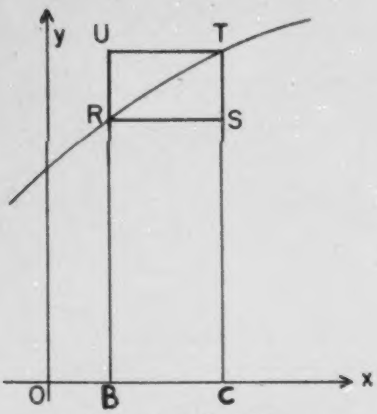


Figure 3-13b.
Graph of a function f.

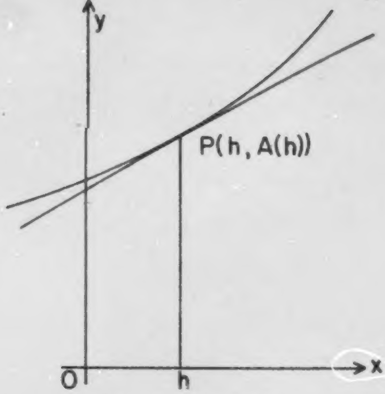


Figure 3-13c.
Graph of the associated area function A.

On the graph of $f: x \longrightarrow f(x)$ (see Figure 3-13c), we shall similarly confine attention to points to the right of $R(h, f(h))$. For definiteness, we assume that the graph of f rises from R to T. Now the area BCTR under the graph is $A(x) - A(h)$. Since this area is greater than that of the rectangle BCSR and less than that of the rectangle BCTU, we have

$$A(x) - A(h) > f(h)(x - h)$$

and $$A(x) - A(h) < f(x)(x - h).$$

Hence $$A(x) > A(h) + f(h)(x - h) \tag{1}$$

and $$A(x) < A(h) + f(x)(x - h). \tag{2}$$

If $x - h$ is small enough, $f(x)$ exceeds $f(h)$ by any artitrarily small amount ϵ. That is, for $x - h$ small enough,

$$f(x) < f(h) + \epsilon .$$

Substituting in (2),

$$A(x) < A(h) + (f(h) + \epsilon)(x - h). \tag{3}$$

Now, (1) and (3) tell us that the graph of A lies between the straight lines

$$y_1 = A(h) + f(h)(x - h)$$

and $$y_2 = A(h) + (f(h) + \epsilon)(x - h)$$

(see Figure 3-13d) for points near enough to P on the right. Since this is true no matter how small ϵ is chosen, we conclude that

$$y = A(h) + f(h)(x - h)$$

is the equation of the required tangent. (Similar considerations apply for points to the left of P.)

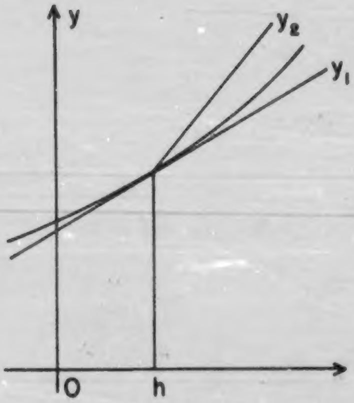


Figure 3-13d.
Linear approximations to the area function.

We assumed that the slope of f was not negative in an interval to the right of (h, f(h)). We could equally well have assumed that the slope was not positive in such an interval; our argument would still hold, with slight modifications which you might investigate. For all the functions that interest us, one or the other of these possibilities must hold, and the argument is therefore quite general. We can then state what we have proved as a theorem.

Theorem. If f is a polynomial function, and has an associated area function A, and if h is any positive number in the domain of f, then A'(h) exists and equals f(h).

Let us turn back to the problem of finding the area function associated with $f: x \longrightarrow x^2$. The theorem makes it unnecessary to carry out the rather lengthy summations of Section 3-12. It is sufficient to find a function A such that

$$A': x \longrightarrow x^2.$$

If A is a polynomial function of degree n, then the degree of A' is n - 1. Going backwards, it is natural to assume that A has the form

$$A: x \longrightarrow ax^3.$$

In this case, $A': x \longrightarrow 3ax^2;$

hence, to obtain $A: x \longrightarrow x^2$

we must choose $a = 1/3$,

and therefore $A\colon x \longrightarrow \frac{1}{3}x^3$

is a possible solution for the required area function.

There are other possibilities of the form

$$A\colon x \longrightarrow \frac{1}{3}x^3 + C, \qquad (1)$$

where C is any constant, since regardless of the choice of C, A' would be given by

$$A'\colon x \longrightarrow x^2.$$

It can be shown that all solutions for A are of the form (1). Now, in fact, we wish to have $A(0) = 0$. This is possible only if $C = 0$, so that our simpler solution was the only correct one.

Let us apply this method to two further examples.

Example 1. Find the area function corresponding to

$$f\colon x \longrightarrow x^3.$$

Solution. Since $A' = f$, we seek an A of the form

$$A\colon x \longrightarrow ax^4.$$

It is easy to show that $a = 1/4$. Moreover the function

$$A\colon x \longrightarrow \frac{1}{4}x^4$$

has a zero at $x = 0$, as it should.

Example 2. Find A if $f\colon x \longrightarrow 2x^2 + 8x - 3$, given that $A(0) = 0$.

Solution. Since $A'\colon x \longrightarrow 2x^2 + 8x - 3$,

by inspection $A\colon x \longrightarrow \frac{2}{3}x^3 + 4x^2 - 3x + C$

is a solution. If $A(0) = 0$, C must be 0. Hence

$$A(x) = \frac{2}{3}x^3 + 4x^2 - 3x.$$

Exercises 3-13

1. Find f' if f is the function

   a) $x \longrightarrow x^2 - x + 3$.

   b) $x \longrightarrow x^2 - x + 18$.

   c) $x \longrightarrow x^4 - 3x^3 + x + 1776$.

   d) $x \longrightarrow x^4 - 3x^3 + x - 1984$.

2. Find two functions g such that g' is the function
   a) $x \longrightarrow 2x$.
   b) $x \longrightarrow 3x^2$
   c) $x \longrightarrow 6x^5 - 3x^2 + 8$
   d) $x \longrightarrow x^4 + x^2 + 1$.
3. For each part of Exercise 2 you obtained two functions. How are these functions related to each other?
4. Find the area bounded by the coordinate axes, the line $x = 2$, and the graph of the function f, where f is
   a) $x \longrightarrow x^2$.
   b) $x \longrightarrow 2x + 1$.
   c) $x \longrightarrow 4x^3 + x$.
5. a) Sketch the graph of $f: x \longrightarrow 3x^2 + 1$.
   b) Mark the region bounded by this graph, the coordinate axes, and the line $x = 1$. Find the area of this region.
   c) Mark the region bounded by your graph, the coordinate axes, and the line $x = 2$. Find the area of this region.
   d) Mark the region bounded by your graph, the x-axis, and the lines $x = 1$ and $x = 2$. How is this region related to the regions you marked in (b) and (c)? Find its area.
6. Find the area bounded by the graph of $f: x \longrightarrow 16 - x^2$, the x-axis, and lines $x = 2$ and $x = 3$. (Hint: see Exercise 5.)
7. Find the area bounded by the graph of $f: x \longrightarrow 4x^3 - x$, the x-axis, and the lines $x = 1$ and $x = 2$.
8. a) Find two different functions g such that $g' = f$, where $f: x \longrightarrow 6x^2 + 2$, and for each of them find the value of $g(2) - g(0)$.
   b) What is the area bounded by the coordinate axes, the graph of $f: x \longrightarrow 6x^2 + 2$, and the line $x = 2$?
9. a) Find two different functions g such that $g' = f$ where $f: x \longrightarrow 4x + 3$, and for each of them find the value of $g(2) - g(1)$.

b) What is the area bounded by the graph of $f: x \longrightarrow 4x + 3$, the x-axis, and the lines $x = 1$ and $x = 2$?

10. If g and h are two different functions such that $g' = h'$, what is the relation between the number $g(5) - g(3)$ and the number $h(5) - h(3)$?

4-15. The Law of Growth. Functional Equations

An avenue through which exponential functions appear in mathematics is the study of various natural phenomena. Physicists observe that radioactive elements decay in such a way that a fixed percentage of the atoms of the element turn into something else in every time interval of fixed length. Suppose that $A(x)$ is the amount of the radioactive element under study that is present at the time x. The amount $A(x)$ can be measured in any convenient units: all the way from the number of individual atoms, up through milligrams, grams, ..., tons (although the last-named unit would be useful only for operations on a very large scale). Now the physicist waits until a time $x + y$ which is later than x and observes the amount $A(x + y)$ of his radioactive element still existing. Since the element decays, there will be less of it. He computes the ratio

$$\frac{A(x + y)}{A(x)}.$$

He may also start at another time u instead of x and then wait for the same additional time. That is, he can observe the amounts $A(u)$ and $A(u + y)$ of his radioactive element at times u and $u + y$. The time intervals from x to $x + y$ and from u to $u + y$ have the same length but they start at different times x and u. The physicist then computes the ratio

$$\frac{A(u + y)}{A(u)}.$$

He will then see that, up to errors in weighing and the like, the ratios are the same:

$$\frac{A(x + y)}{A(x)} = \frac{A(u + y)}{A(u)}. \qquad (1)$$

Exactly the same sort of thing, except for growth instead of decay, is observed by biologists in the increase of bacterial population under suitable conditions of temperature, food supply, and living space. Suppose that $N(x)$ represents the number of bacteria in the colony at time x. Counting the little creatures at time $x + y$, the biologist finds that their number has increased: $N(x + y)$ is greater than $N(x)$. He computes the ratio

$$\frac{N(x + y)}{N(x)}.$$

Counting his bacteria at another time u and then at time $u + y$ (just as far beyond u as $x + y$ is beyond x), he gets population counts $N(u)$ and $N(u + y)$. Computing the ratio

$$\frac{N(u + y)}{N(u)},$$

he finds that it is the same as the first ratio:

$$\frac{N(x + y)}{N(x)} = \frac{N(u + y)}{N(u)}. \tag{2}$$

This law of growth is, as a matter of fact, fairly rare among actual populations of bacteria, animals, and so on, but it is found in some cases to be a good approximation to what actually happens.

You will note that the Equations (1) and (2) for the functions A and N are exactly the same. The only difference is that A decreases as time increases and N increases as time increases. Even this is not reflected in the Equations (1) and (2).

It is easy to see that there is some connection between exponentials and functions N that satisfy (2). To take a simple case, suppose that when his experiment starts there are N_0 bacteria in the colony and that the colony doubles in size in a unit of time, which might be an hour, a day, a week, etc. (The unit used is of no consequence in our present argument). We have then

$$\frac{N(1)}{N(0)} = 2 = \frac{N(1)}{N_0}$$

if we start the experiment at time zero. Thus $N(1) = 2N_0$.

The law of growth (2) tells us that

$$\frac{N(2)}{N(1)} = \frac{N(1)}{N(0)},$$

and thus $N(2) = \frac{(N(1))^2}{N(0)}$ or $N(2) = 4N_0$.

Proceeding in this way, we see that

$$N(3) = 8N_0,$$
$$N(4) = 16N_0,$$
$$\cdots\cdots\cdots\cdots$$
$$N(k) = 2^k N_0$$

for every positive integer k. Thus there is certainly a connection between the law of growth expressed in (2) and powers of numbers.

It turns out that functions satisfying laws of growth or decay like (2) are exactly exponential functions. They are not polynomials, or even remotely like polynomials, but form an entirely new class of functions. In this section, we shall study functions with this particular growth or decay property; that is, we shall write down in precise terms what our growth phenomenon means and then study the function in terms only of this growth phenomenon. Our attitude is that of the naturalist who wishes to identify or reconstruct an entire animal from the skeleton alone.

We first derive an equation which our functions must satisfy. Suppose that we have a function g, defined for all real numbers, for which the proportional increase is the same in intervals of equal length. To state this precisely, let x, y, and u be any real numbers. Let us look at the values of g at the points x, $x + y$, u, and $u + y$. We obtain the numbers $g(x)$, $g(x + y)$, $g(u)$, and $g(u + y)$. The intervals on the real line from x to $x + y$ and from u to $u + y$ both have the same length, namely, y. Thus the proportionate increase of g will be the same in these two intervals (see Figure 4-15a).

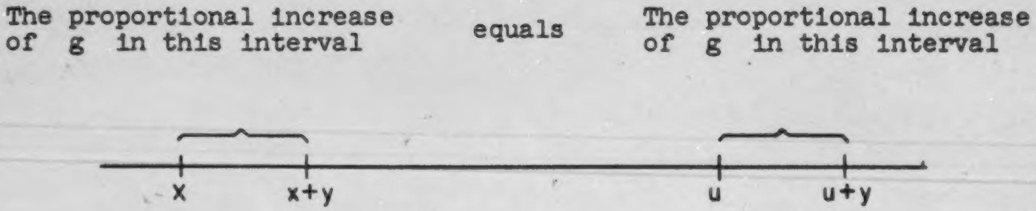

The proportional increase of g in this interval
equals
The proportional increase of g in this interval
x
x+y
u
u+y

Figure 4-15a

This means simply that

$$\frac{g(x + y)}{g(x)} = \frac{g(u + y)}{g(u)}. \qquad (3)$$

We repeat that the Equation (3), which is called a functional equation and which describes the behavior of the function g, is to hold for all possible choices x, y, and u. The Equation (3) has meaning only if g assumes the value 0 nowhere.

We are going to tinker with the equation (3) and show what the function g will have to be. The trick here is that we can choose x, y, and u in any way we please. This will produce surprising results.

Our first maneuver is to leave x and y arbitrary and to set $u = 0$. This means merely that we will begin one of our test intervals at 0. The Equation (3) then becomes

$$\frac{g(x + y)}{g(x)} = \frac{g(y)}{g(0)}.$$

Multiply both sides of this equality by $g(x)$. This gives

$$g(x + y) = \frac{g(x)g(y)}{g(0)}. \qquad (4)$$

Equation (4) is still a little clumsy. It will help matters to divide both sides of (4) by the number $g(0)$. (Recall that g cannot be zero anywhere if (3) is to hold.) Doing this, we get

$$\frac{g(x + y)}{g(0)} = \frac{g(x)}{g(0)} \cdot \frac{g(y)}{g(0)}. \qquad (5)$$

We now look at the function $f\colon x \longrightarrow \frac{g(x)}{g(0)}$. This function f is just as unknown a function as g, but it satisfies a simpler functional equation that does g. If we replace $\frac{g}{g(0)}$ by f in (5), we obtain the functional equation

$$f(x + y) = f(x)f(y), \tag{6}$$

which must hold for all real numbers x and y.

We re-state what we have done so far. If g is a function that satisfies (3), then $\frac{g}{g(0)}$ is a function that satisfies the functional equation (6).

Exercises 4-15a

1. Prove that the function $f\colon x \longrightarrow 0$ satisfies the functional equation (6). Does this function satisfy the functional equation (3)? If not, why not? Can you explain how it happens that there are functions satisfying (6) that do not satisfy (3)?
2. Prove that the function $f\colon x \longrightarrow 1$ satisfies the functional equation (6). Does this function satisfy the functional equation (3)?
3. Let f be a solution of the functional equation (6) that is nowhere zero and let A be any non-zero real number. Prove that the function $g\colon x \longrightarrow Af(x)$ is a solution of the functional equation (3).

Let us now state our program. We want to determine an unknown function f, about which we know only that it satisfies the functional equation

$$f(x + y) = f(x)f(y) \tag{6}$$

for all $x, y \in R$. There are two trivial solutions of (6) for which f is a constant function, namely $f\colon x \longrightarrow 0$ and $f\colon x \longrightarrow 1$. In order to eliminate these two trivial cases, and also to obtain considerably more useful results, we shall impose on f the further condition that it either is strictly increasing or is strictly decreasing.

We have eliminated the two trivial constant solutions of (6). Aside from these, there are no polynomial functions that will satisfy (6). In fact, let $f: x \longrightarrow a_0 + a_1x + \dots + a_nx^n$, $a_n \neq 0$, be any polynomial function of degree $n > 0$. Then if f satisfies the functional equation (6), we would have in particular

$$f(x + x) = f(2x) = f(x)f(x) = \big(f(x)\big)^2$$

for all real numbers x. Now $f(2x)$ is a polynomial of degree n, like $f(x)$ itself, while $\big(f(x)\big)^2$ is a polynomial of degree $2n$. Since two polynomial functions are equal for all real numbers, if and only if they have the same degree and the same coefficients for each power of x appearing in the polynomials, we see that $f(2x) = \big(f(x)\big)^2$ is an identity impossible to satisfy for all x if $n > 0$.

Exercises 4-15b

1. Prove that the polynomial $f: x \longrightarrow mx + b$ is not a solution of (6) if $m \neq 0$.
2. Find real numbers x and y such that $(x + y)^2 \neq x^2y^2$. Prove from this that $f: x \longrightarrow x^2$ is not a solution of the functional equation (6).

We have thus seen that the functional equation (6) has no familiar functions as its solutions except for the trivial solutions $f: x \longrightarrow 0$ and $f: x \longrightarrow 1$. We return, then, to our problem. We have an unknown function, defined for all real numbers, that satisfies Equation (6). Since (6) is supposed to be satisfied for all x and y, we can choose x and y in any manner we wish, and draw all of the consequences we can from clever (or lucky) choices.

Our first choice is to take as x some number a for which $f(a) \neq 0$. There is such a number, since f is not a constant function. We take $y = 0$. Doing this, we get

$$f(a + 0) = f(a)f(0).$$

Since $a + 0 = a$,

$$f(a) = f(a)f(0).$$

Dividing by $f(a)$ $(\neq 0)$, we find

$$f(0) = 1, \tag{7}$$

which is a nice first bit of information about f.

Our next selection of x and y is to keep x arbitrary and to set $y = -x$. This gives

$$f\big(x + (-x)\big) = f(x)f(-x),$$

or

$$f(0) = f(x)f(-x).$$

In view of (7), this may be rewritten as

$$f(x)f(-x) = 1. \tag{8}$$

Exercises 4-15c

1. Use the Identity (8) to prove that the function f is zero for no value of x.
2. Prove from the Identity (8) that

$$f(-x) = \frac{1}{f(x)}$$

for all real numbers x.

Exercise 2 in the preceding set shows that we can find $f(x)$ for all real x if we know $f(x)$ for positive values of x. (If x is negative, then $f(x)$ is the reciprocal of $f(-x)$, where $-x$ is positive.) We sketch the situation in Figure 4-15b.

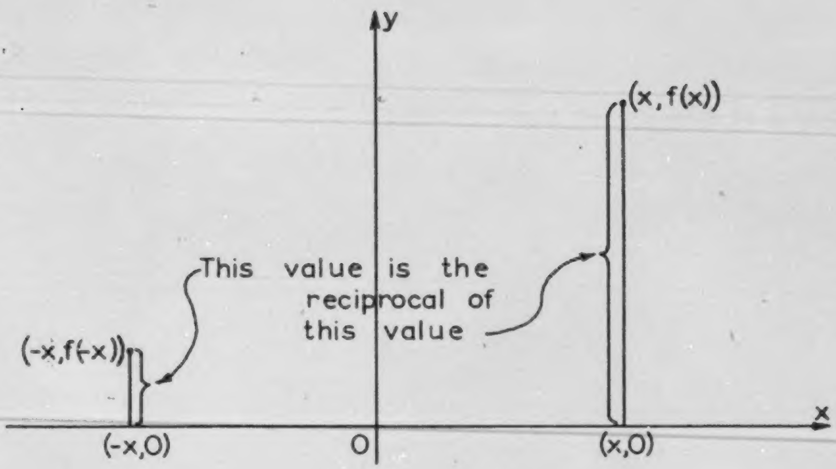


Figure 4-15b

We now observe a curious fact about f. Given any number x, we can write x as $\frac{1}{2}x + \frac{1}{2}x$. The functional equation (6) then shows that

$$f(x) = f(\frac{x}{2} + \frac{x}{2}) = f(\frac{x}{2})f(\frac{x}{2}) = [f(\frac{x}{2})]^2. \qquad (9)$$

We see from Exercise 1 above that $f(\frac{x}{2})$ is different from zero. We know that the square of any non-zero real number is a positive real number. The Equality (9) therefore proves that f(x) is greater than zero for all real numbers x.

This observation is important, and we shall come back to it shortly.

We have already seen that when x = y, the functional equation (6) gives us

$$f(x + x) = f(2x) = f(x)f(x) = [f(x)]^2.$$

This is a very important step in the development, and the following exercises lead to a generalization of it.

Exercises 4-15d

1. By setting $y = 2x$ in the functional equation (6), prove that $f(3x) = [f(x)]^3$ for all real numbers x.

2. Prove by finite induction that

$$f(mx) = [f(x)]^m \tag{10}$$

for all positive integers m and real numbers x.

Now in the Identity (10), let us make the substitution $x = \frac{y}{m}$, where y is any real number. This gives

$$f(m \cdot \tfrac{y}{m}) = [f(\tfrac{y}{m})]^m.$$

Simplifying and writing x for y, we get

$$f(x) = [f(\tfrac{x}{m})]^m, \tag{11}$$

true for all real numbers x and positive integers m.

Let us see what the Identity (11) means. It means that we find $f(x)$ from $f(\frac{x}{m})$ by raising $f(\frac{x}{m})$ to the m-th power. Turning this around, we can find $f(\frac{x}{m})$ from $f(x)$ by taking the positive m-th root of $f(x)$. That is,

$f(\frac{x}{m})$ _is the unique positive number whose m-th power is $f(x)$._

We write the last sentence in symbols as

$$f(\tfrac{x}{m}) = [f(x)]^{\frac{1}{m}}. \tag{12}$$

Now put nx for x in the Identity (12), where n is a positive integer. This gives

$$f(\tfrac{nx}{m}) = [f(nx)]^{\frac{1}{m}}.$$

We already know that $f(nx) = [f(x)]^n$ $\big($see (10)$\big)$, and therefore the last equation can be re-written as

$$f(\frac{nx}{m}) = [f(x)]^{\frac{n}{m}}.$$

To save a little writing, we replace the quotient $\frac{n}{m}$ in this last identity by the symbol r. Thus we have

$$f(rx) = [f(x)]^r \tag{13}$$

for all real numbers x and all positive rational numbers r.

It is now time to take stock of where our reconstruction of the function f has brought us. We select any convenient non-zero number for x, in the Identity (13). The number 1 has obvious advantages, and so we take it. For brevity, we write a for the number f(1). (Remember that a is positive.) Then, what does our Identity (13) tell us? We can get some idea by choosing some specific values for r.

Putting $r = 1, 2, 3, 4, 5, 6, \ldots$ we write

$$\begin{aligned} f(1) &= a \\ f(2) &= a^2 \\ f(3) &= a^3 \\ f(4) &= a^4 \\ f(5) &= a^5 \\ f(6) &= a^6 \\ &\vdots \end{aligned}$$

Putting $r = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \ldots,$ we write

$$\begin{aligned} f(\tfrac{1}{2}) &= a^{\frac{1}{2}} \\ f(\tfrac{1}{3}) &= a^{\frac{1}{3}} \end{aligned}$$

and so on.

Exercises 4-15e

1. Find the numbers $f(\frac{1}{3})$, $f(\frac{1}{4})$, $f(\frac{3}{4})$, $f(\frac{1}{5})$, $f(\frac{2}{5})$, $f(\frac{3}{5})$, and $f(\frac{4}{5})$ in terms of the number $a = f(1)$.
2. Find the number $f(\frac{371}{1000})$ in terms of $a = f(1)$.
3. Suppose that $f(1) = 1$. Find $f(r)$ for all positive rational numbers r.
4. Suppose that $f(a) = 1$ for some positive rational number a. Find $f(r)$ for all positive rational numbers r.

We now sketch the graph of the function f for non-negative rational values, in Figure 4-15c. Notice that the three cases $a > 1$, $a = 1$, and $0 < a < 1$ lead to quite different graphs.

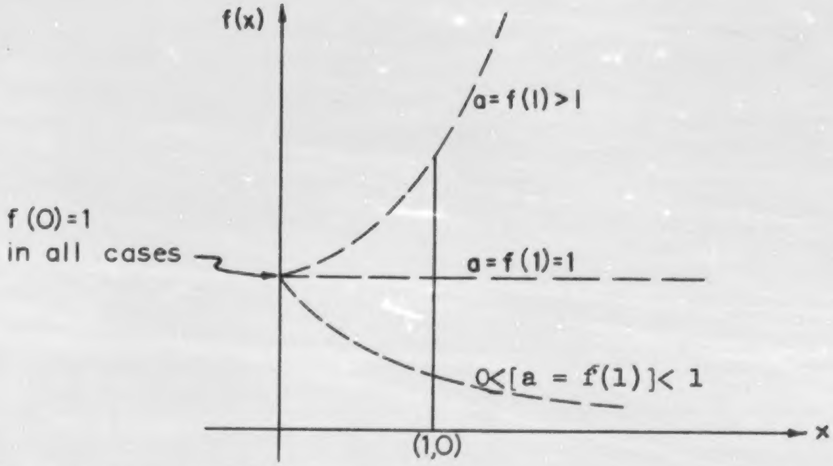


Figure 4-15c.

It is not hard to see that our graphs have the form shown in Figure 4-15c. Suppose that $a > 1$. Then if r and s are rational numbers such that $0 \leq r < s$, we have $a^r < a^s$; that is, the function $f: r \longrightarrow a^r$ is a strictly increasing function of r, for non-negative rational numbers r. For $a = 1$, we get $a^r = 1^r = 1$ for all non-negative rational numbers r. For $0 < a < 1$, we have $a^r > a^s$ if $0 \leq r < s$ and r and s are rational.

We can now extend our graph of the function f to negative rational values, by use of the relation $f(-x) = \frac{1}{f(x)}$. The case $a = f(1) > 1$ is sketched in Figure 4-15d.

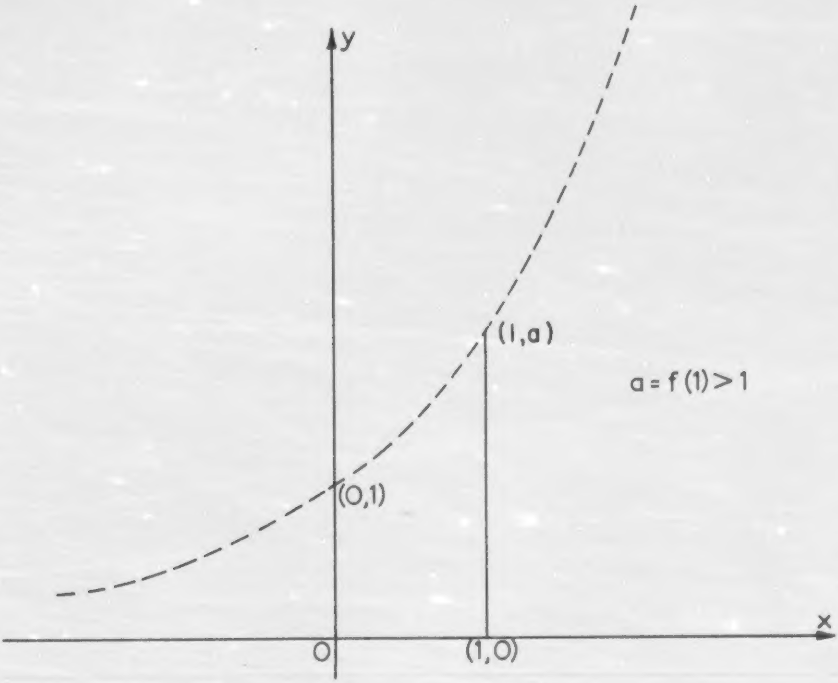


Figure 4-15d.

Exercises 4-15f

1. Draw a graph of the function f for all rational values if $f(1) = a = 1$.
2. Draw a graph of the function f for all rational values if $f(1) = a < 1$.

We have now the problem of filling in the values $f(x)$ for irrational values of x. Unfortunately, the functional equation (6) does us no good at all here. One can show, for example, that there are functions f such that $f(x + y) = f(x)f(y)$ and such that $f(1)$ and $f(\sqrt{2})$ are any two positive numbers you like.

At this point, our requirement that f be either an increasing or a decreasing function comes to our rescue. With its aid, the function $f: r \longrightarrow a^r$, defined for all rational numbers r, can be extended in a unique way to a function $f: x \longrightarrow a^x$ defined for all real numbers x. Because this extension is essentially the same as that given in Section 4-3 for the special case $a = 2$, we shall not repeat it here.

Exercises 4-15g

1. What well-known function f satisfies the functional equation

$$f(xy) = f(x) + f(y)$$

for all positive real numbers x and y?

2. Consider the function f, defined for all real x and y, which satisfies the functional equation

$$f(x + y) = f(x) + f(y). \tag{A}$$

   a. By setting $x = 1$ and $y = 0$ in (A), show that

$$f(0) = 0. \tag{B}$$

b. By setting $y = -x$ in (A), and using (B), show that

$$f(-x) = -f(x). \tag{C}$$

c. By setting $y = x$ in (A) show that $f(2x) = 2f(x)$. What does this imply for $f(3x)$? $f(4x)$? Using mathematical induction, show that

$$f(mx) = mf(x) \tag{D}$$

for any natural number m.

d. Replace x by $\frac{x}{m}$ in (D) and thus show that

$$f(\tfrac{x}{m}) = \tfrac{1}{m}f(x) \tag{E}$$

for any natural number m.

e. Combine (D) and (E) to show that

$$f(rx) = rf(x) \tag{F}$$

for any rational number $r > 0$.

f. With the aid of (B) and (C), show that (F) is also valid for rational $r \leq 0$.

g. Now write a for $f(1)$ and show that

$$f(r) = ar \tag{G}$$

for all rational numbers r.

*h. Now add the assumption that f is increasing or is decreasing, and show that f must then be

$$f: x \longrightarrow ax \qquad (a \neq 0)$$

for all real numbers x.

3. One function f, such that, for some real number a,

$$f(x + y) = f(x) + ay$$

for all real numbers x and y, is clearly the function $f: x \longrightarrow ax$ of Exercise 2. Find another.

4-16. An Approximation for e^x.

In Section 4-12, we wrote a succession of approximations to e^x for $|x|$ small, namely:

$$e^x \approx 1 + x \tag{1}$$

$$e^x \approx 1 + x + \frac{x^2}{2} \tag{2}$$

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} \tag{3}$$

$$\cdots$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \tag{n}$$

Although we showed by graphs and by sample computations that satisfactory approximations were obtained in this way, we did not explain how the results (1) to (n) were obtained, nor did we give an estimate of the error made in using the replacement indicated. In this section we shall discuss both of these questions.

It is, of course, impossible to find a polynomial function g whose slope function g' is the same as g, as is the case for $f: x \longrightarrow e^x$. One good reason is that the degree of g' is necessarily one less than the degree of g. Whatever approximating polynomial is chosen, this will be the case.

We already know the best approximating polynomial of first degree, namely

$$1 + x.$$

In seeking a second degree polynomial $g_2(x)$ whose values will approximate e^x for $|x|$ small, we wish the graph of g_2 to have the same tangent, $y = 1 + x$, at the point $(0,1)$ as the graph of $x \longrightarrow e^x$. Accordingly, we assume that

$$g_2(x) = 1 + x + a_2x^2.$$

How shall we choose a_2?

The slope $g_2'(x) = 1 + 2a_2x$ cannot be made to agree with $g_2(x)$. But we can make it agree with our best first degree choice,

$$g_1(x) = 1 + x.$$

To do this, we set

$$1 + 2a_2x = 1 + x$$

and obtain

$$a_2 = \frac{1}{2}.$$

Thus

$$g_2(x) = 1 + x + \frac{1}{2}x^2, \tag{2}$$

with

$$g_2'(x) = 1 + x,$$

is our suggested best second degree approximation. Without inquiring for the present whether this is indeed the best such approximation and if so, how good it is, let us continue the process.

If we seek a third degree polynomial, we preserve the measure of approximation already achieved and write

$$g_3(x) = 1 + x + \frac{1}{2}x^2 + a_3x^3.$$

We determine a_3 so that

$$g_3'(x) = 1 + x + 3a_3x^2$$

agrees with

$$g_2(x) = 1 + x + \frac{1}{2}x^2,$$

our best second degree approximation. This gives $a_3 = \frac{1}{3\cdot 2}$, and

$$g_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\cdot 3}x^3. \tag{3}$$

In the same way, we can build up by induction

$$g_n(x) = 1 + x + \frac{1}{2}x^2 + \dots + \frac{x^n}{n!}. \tag{n}$$

This achieves our first goal, the motivation for the choice of these particular polynomials.

What about the error made in replacing e^x by one of these expressions? In answering this question we shall confine ourselves to positive values of x.

The first observation is that any one of these approximations, $g_n(x)$, $n = 1, 2, 3, \ldots,$ is too small. Since

$$g_n'(x) = g_n(x) - \frac{x^n}{n!},$$

$$g_n'(x) < g_n(x) \qquad \text{when } x > 0.$$

The graph of the function $f\colon x \longrightarrow e^x$ climbs at such a rate that $f'(x)$ is always equal to $f(x)$. Since the graph of $g_n(x)$ climbs less rapidly, it will fall below that of f and $g_n(x)$ will be too small.

To estimate the error, we need an approximation which is known to be too large. Then the true value e^x will be known to lie between two estimates, one of which is too small and the other too large. We shall illustrate with polynomials of third degree.

As we know,

$$g_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

gives too small a value for e^x, $x > 0$. Now consider

$$h_3(x) = 1 + x + \frac{x^2}{2!} + \frac{cx^3}{3!}$$

where $c > 1$. Certainly $h_3(x) > g_3(x)$. It turns out that for sufficiently small positive values of x, $h_3(x)$ is also greater than e^x, as we now show. We wish to have $h_3(x)$ climb too fast to represent $x \longrightarrow e^x$. This will be true if the slope is greater than the ordinate, that is,

$$h_3'(x) > h_3(x). \qquad \text{(A)}$$

Since $h_3'(x) = 1 + x + \frac{cx^2}{2!}$,

(A) will hold when

$$\frac{cx^2}{2!} > \frac{x^2}{2!} + \frac{cx^3}{3!},$$

that is, when

$$(c - 1)\frac{x^2}{2!} > \frac{cx^3}{3!}.$$

Dividing by $\frac{x^2}{2!}$, we have

$$c - 1 > \frac{cx}{3}$$

or

$$x < 3\left(\frac{c - 1}{c}\right). \qquad \text{(B)}$$

Hence for positive values of x less than $3(\frac{c - 1}{c})$, $h_3(x)$ gives too large a value for e^x.

To be concrete, we take $c = 2$. Then

$$e^x < 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!}$$

if $0 < x < \frac{3}{2}$.

Of course, in the same interval,

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

If, for example, we wish to compute $e^{.2}$, we know that

$$e^{.2} < 1 + .2 + \frac{.04}{2} + \frac{.008}{3} = 1.222666\cdots$$

and

$$e^{.2} > 1 + .2 + \frac{.04}{2} + \frac{.008}{6} = 1.221333\cdots$$

The same principle applies to approximations of n^{th} degree and gives the result

$$e^x < 1 + x + \frac{x^2}{2!} + \ldots + \frac{cx^n}{n!}, \qquad c > 1,$$

for

$$0 < x < n\left(\frac{c - 1}{c}\right). \qquad \text{(C)}$$

We now show that

$$g_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

is the best possible n^{th} degree polynomial for approximating e^x for x arbitrarily near 0. Let

$$h_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{cx^n}{n!}, \qquad c > 1.$$

Choose c as close to 1 as you please. Then the graph of $x \longrightarrow e^x$ lies between those of g_n and h_n for $0 < x < n(\frac{c-1}{c})$. Hence, no choice of $c \neq 1$ in $h_n(x)$ could improve on $g_n(x)$, which corresponds to $c = 1$. The student should convince himself that changes in the coefficients of the lower degree terms would also effect no improvement.

Exercises 4-16

1. Compute $e^{0.01}$ correct to five places of decimals. Obtain the value of each term to six places, continuing until you reach terms which have only zeros in the first six places, add, and round off to five places. How many terms did you need to use? Note that even though the remaining terms are individually less than 0.000001, they might accumulate to give a very large sum; in this particular case, they do not.
2. Carry out the argument in the text to obtain $g_4(x)$ and $g_5(x)$.
3. Parallel the argument which leads to (B), to establish (C).
4. Calculate upper and lower approximations to $e = e^1$, using $g_5(x)$ and $h_5(x)$ with $c = 2$.

4-17. Computation of e^x by the Use of the Area Function.

If the student has studied Sections 2-8 and 2-9 in the appendix to Chapter 2, he will have the background to understand a powerful method for computing values of $f: x \longrightarrow e^x$.

In these Sections, $A(x)$ was used to represent the area below the graph of f, above the x-axis and bounded on the left and right by vertical lines at 0 and x (see Figure 4-17a).

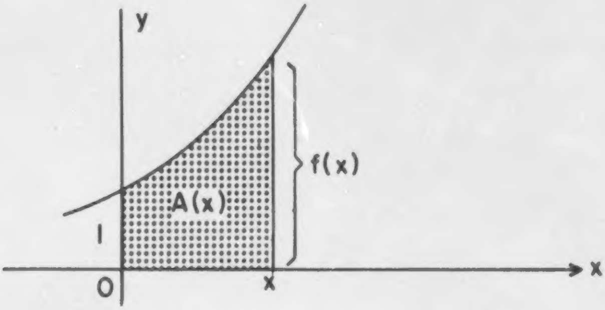


Figure 4-17a. Area below the graph of $f: x \longrightarrow e^x$.

It was proved that for a continuous function $f: x \longrightarrow f(x)$

$$A'(x) = f(x). \qquad (1)$$

In the case of $f: x \longrightarrow e^x$ we also know (see Section 4-6) that

$$f'(x) = f(x). \qquad (2)$$

It follows from (1) and (2) that, in this instance,

$$f'(x) = A'(x).$$

At any given value of x, the graphs of f and A have the same slope and therefore parallel tangents (see Figure 4-17b).

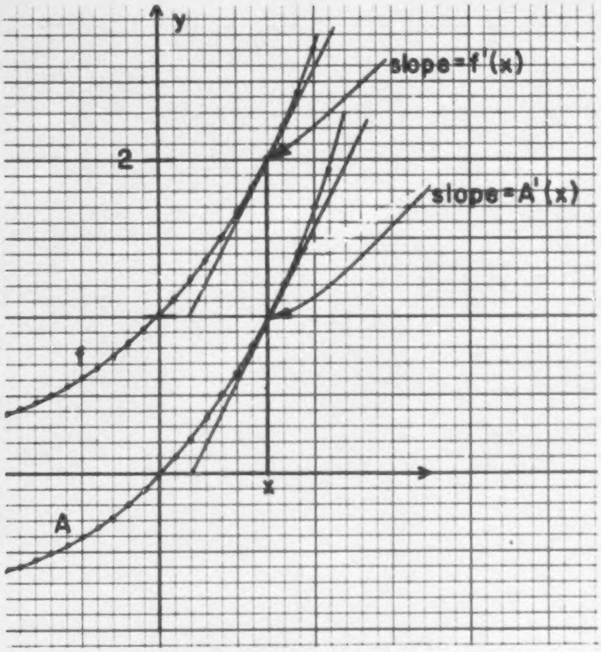


Figure 4-17b.
The graphs of $x \longrightarrow f(x)$ and $x \longrightarrow A(x)$.

The graphs are not identical since $A(0) = 0$ but $f(0) = 1$. The relation between $f(x)$ and $A(x)$ is given by

$$f(x) = 1 + A(x). \tag{3}$$

Equation (3) is fundamental to our new approach. Since the graph of f lies above the tangent $y = 1 + x$ to the right of $P(0,1)$ (see Figure 4-17c), $A(x)$ is greater than the area of the trapezoid OATP.

Hence, $A(x) > OA \cdot \frac{OP + AT}{2}$,

that is,

$$A(x) > x \cdot \frac{1 + (1 + x)}{2}$$
$$= x + \frac{x^2}{2}, \qquad x > 0.$$

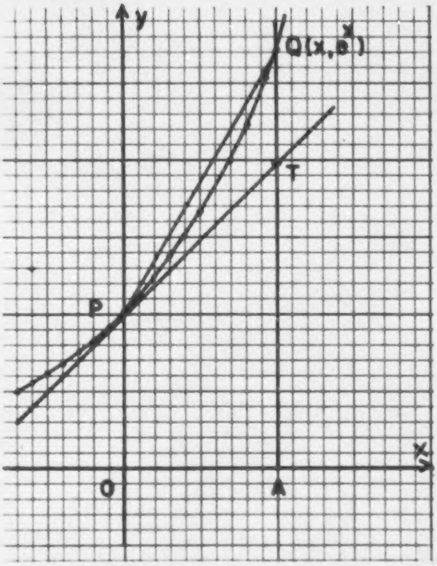


Figure 4-17c.
Estimating $A(x)$.

But $e^x = f(x) = 1 + A(x)$.

Hence
$$e^x > 1 + x + \frac{x^2}{2}. \qquad (4)$$

On the other hand, since the graph is concave upward, it lies below the chord PQ. Hence, $A(x)$ is less than the area of trapezoid OAQP and

$$A(x) < OA \cdot \frac{OP + AQ}{2}.$$

That is,
$$A(x) < x\left(\frac{1 + e^x}{2}\right), \qquad x > 0.$$

Since
$$A(x) = f(x) - 1 = e^x - 1,$$
$$e^x - 1 < x\left(\frac{1 + e^x}{2}\right), \qquad (5)$$

which simplifies to

$$e^x < \frac{2 + x}{2 - x}, \qquad \text{for} \quad 0 < x < 2. \qquad (6)$$

We may combine the Inequalities (4) and (6) to obtain

$$1 + x + \frac{x^2}{2} < e^x < \frac{2 + x}{2 - x}, \qquad 0 < x < 2. \qquad (7)$$

Let us use the Inequalities (4) and (6) to compute $e^{0.1}$.

By (4) $\qquad e^{0.1} > 1 + .1 + \frac{.01}{2} = 1.1050.$

By (6) $\qquad e^{0.1} < \frac{2.1}{1.9} = 1 + \frac{2}{19} \approx 1.10526.$

We may therefore use 1.105 as the value of $e^{0.1}$ to 3 decimal place accuracy.

From this result for $e^{0.1}$, it is easy to compute $e^{0.2}$, $e^{0.4}$, and $e^{0.8}$. From these results, in turn, we may find an approximation to e. (See Exercise 1.)

Exercises 4-17

1. Obtain an approximation to e by computing successively $e^{0.2} = (e^{0.1})^2$, $e^{0.4}$, $e^{0.8}$, $e = (e^{0.2})(e^{0.8})$.

2. Obtain an approximation to e by using a table of logarithms to calculate $(e^{0.1})^{10}$. Compare your result with that obtained in Exercise 1.

3. Compute $e^{1.2}$ in two different ways using only the results obtained in this section. Compare your answers with entries in Table 4-6.

4-18. An Approximation for ln x.

As we know from Section 4-12,

$$\ln x \approx (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3}$$

for values of x near 1. It is convenient to let $x - 1 = u$ so that

$$\ln (1 + u) \approx u - \frac{u^2}{2} + \frac{u^3}{3} \quad \text{for } |u| \text{ small.}$$

As you might guess, better approximations are given by

$$u - \frac{u^2}{2} + \frac{u^3}{3} - \ldots \pm \frac{u^n}{n} \qquad \text{for } n > 3 \tag{1}$$

where the $+$ sign is chosen if n is odd and the $-$ sign if n is even.

When u is positive (so that $x > 1$), the terms of (1) are alternately positive and negative. It can be shown that in this case the error E made in replacing $\ln(1 + u)$ by (1) is numerically less than $\frac{u^{n+1}}{n+1}$. Specifically, if n is odd so that the last term of (1) is $+\frac{u^n}{n}$, the correct value of $\ln(1 + u)$ is less than

$$u - \frac{u^2}{2} + \frac{u^3}{3} - \ldots + \frac{u^n}{n}$$

but greater than

$$u - \frac{u^2}{2} + \frac{u^3}{3} - \ldots + \frac{u^n}{n} - \frac{u^{n+1}}{n+1}.$$

A similar statement can be made if n is even.

In the light of these remarks about error, we can be sure, for example, that

$$\begin{aligned} \ln 1.1 &= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - E \\ &= 0.1 - 0.005 + 0.00033\ldots - E \\ &\approx 0.095333 - E, \end{aligned}$$

where E is positive but less than $\frac{0.0001}{4} = 0.000025$. The correct value of $\ln 1.1$ to 5 decimal places is 0.09531.

If we examine the expression which represents an upper bound for the error E, we note that if $u < 1$, $\frac{u^{n+1}}{n+1}$ can be made arbitrarily small by choosing n large enough. That is, a polynomial approximation can be found which is as accurate as you please. This is no longer true if $u > 1$. In fact, in this case, $\frac{u^{n+1}}{n+1}$ can be made arbitrarily large by choosing n large enough. Hence if $u > 1$, we cannot use (1) to give an arbitrarily good approximation to $\ln(1 + u)$, no matter how large an n we take.

The approximation (1) can be used also for $-1 < u < 0$. In this case, all the terms have negative values. If we write

$$u = -v, \quad \text{(1) becomes}$$

$$\ln(1 - v) \approx -(v + \frac{v^2}{2} + \frac{v^3}{3} + \dots + \frac{v^n}{n}), \quad 0 < v < 1. \qquad (2)$$

For example, with $v = 0.1$ and $n = 4$,

$$\ln(1 - .1) = \ln 0.9 \approx -(.1 + \frac{.01}{2} + \frac{.001}{3} + \frac{.0001}{4}) \approx -0.10536.$$

The correct value, to 5 decimal places, is -0.10536. Of course, we may write

$$\ln 0.9 = \ln \frac{9}{10} = -\ln \frac{10}{9} = -\ln(1 + \frac{1}{9})$$

and apply (1) with $u = \frac{1}{9}$. This procedure has the advantage that we are in a position to estimate the maximum error made in using the polynomial approximation. In the case of (2), the signs do not alternate and the estimation of error is more difficult.

In practice, a number of short-cuts are used to reduce the amount of computation necessary. These refinements do not concern us since we have restricted ourselves to essentials.

A fuller justification of the methods discussed depends upon a knowledge of the calculus.

Exercises 4-18

1. Use (1) with $n = 4$ to calculate ln 1.2. Estimate the maximum error.

2. Show that if (1) is used to calculate ln 2, it is necessary to use 1000 terms to assure an accuracy of .001 and that therefore (1) gives an unsatisfactory approximation. Note that here $u = 1$. Why would you expect some difficulty here?

3. Find ln .8, using the scheme applied to ln .9.

4. From ln .8 and ln .9, find ln .72.

5. Find ln 1.44 from ln 1.2.

6. From ln .72 and ln 1.44 find ln 2.

7. Show that the device used to handle ln .9 in terms of $\ln \frac{10}{9}$ fails for $\ln x$ if $0 < x < 0.5$. How could you find ln .25 without using the approximation (2) directly?

Appendices

Chapter 5

5-15. The Measurement of Triangles.

This Section is a brief outline of elementary trigonometry, included for the benefit of those students who are not familiar with this topic or who may wish to review it.

We introduced, in Section 5-3, the notions of degree measure and of the circular (or trigonometric) functions of angles. Because degree measure is traditional in elementary trigonometry, we shall use it exclusively in the rest of this Section. Thus, if an angle measures x°, we refer to Table III (page 300) and find, as appropriate, $\sin^{\circ} x$, $\cos^{\circ} x$ or $\tan^{\circ} x$. For economy of notation, however, we shall omit the degree sign and write merely $\sin x$, $\cos x$ or $\tan x$.

If θ is the angle from a ray R_1 to a ray R_2, then we say θ is in standard position if its vertex is at the origin of the uv-plane and R_1 extends along the positive u-axis. If R_2 meets the unit circle $u^2 + v^2 = 1$ at a point Q, then the coordinates of Q are $(\cos \theta, \sin \theta)$. If, moreover, $P(u,v)$ is a point r units from the origin on R_2, we have, by similar triangles (see Figure 5-15a) and by the definition of tan,

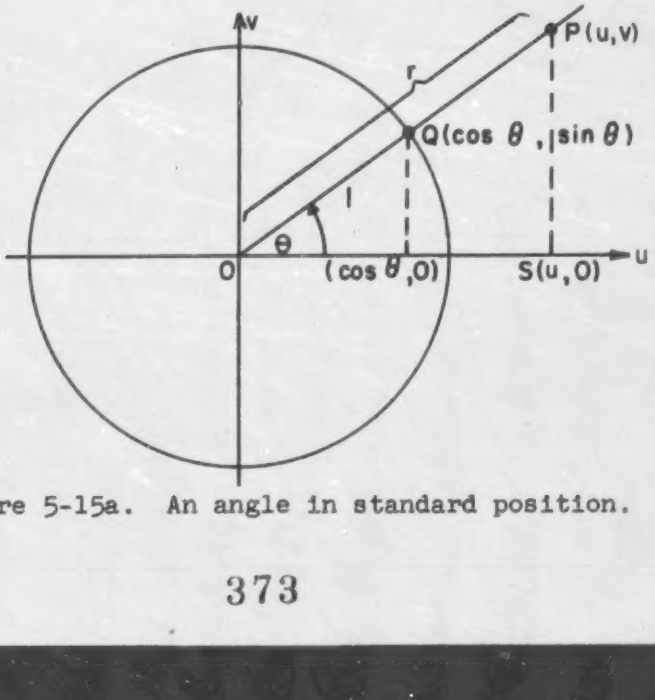


Figure 5-15a. An angle in standard position.

Theorem 5-15a: $\frac{u}{r} = \cos\theta$, $\frac{v}{r} = \sin\theta$, $\frac{v}{u} = \tan\theta$, $(u \neq 0)$.

We are now ready to discuss some examples of a simple and important application of the trigonometric functions -- the indirect measurement of distances by triangulation.

Example 1. At a point 439 feet from the base of a building the angle between the horizontal and the line to the top of the building is 31°. What is the height of the building?

Solution: Let P be the point with abscissa 439 on the terminal side of the angle 31° in standard position (Figure 5-15b). Then the ordinate v of P is the height of the building. By Theorem 5-15a

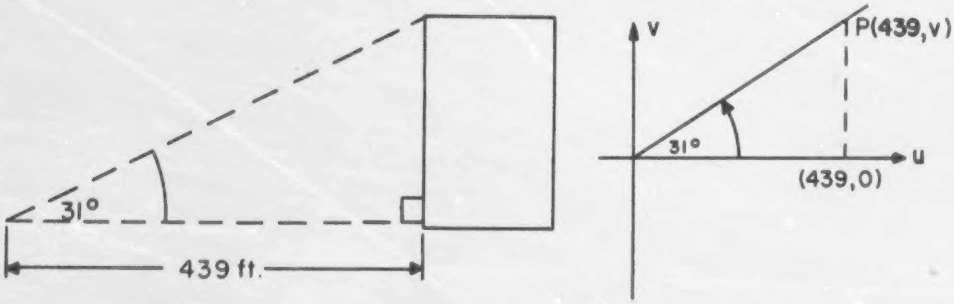


Figure 5-15b.

$$\frac{v}{439} = \tan 31^\circ$$

and from Table III

$$\tan 31^\circ \approx .601.$$

We have

$$\frac{v}{439} \approx .601$$

$$v \approx 439(.601) \approx 264$$

so that the building is approximately 264 feet high.

Example 2. To measure the width of a river a stake was driven into the ground on the south bank directly south of a tree on the opposite bank. From a point 100 feet due west of the stake the tree was sighted and the angle between the line of sight and the east-west line measured. What is the width of the river if this angle was 60°?

Solution: The point from which the tree was sighted was taken due west of the stake so that the angle RST (Figure 5-15c) would be a right angle. Let P be the point with abscissa 100 on the terminal side of the angle 60° in standard position.

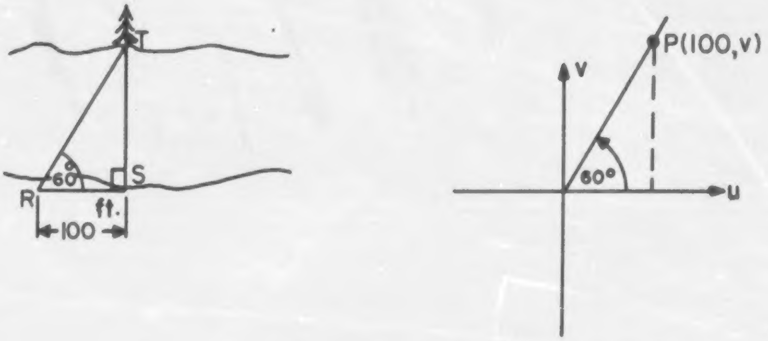


Figure 5-15c.

Then the ordinate v of P will be the width of the river. From Theorem 5-15a and Table III we have

$$\frac{v}{100} = \tan 60^\circ \approx 1.732$$

$$v \approx 100(1.732) \approx 173.2.$$

The river is approximately 173 feet wide.

Example 3. At the instant when the moon is exactly at half phase the angle between the line from the earth to the moon and the line from the earth to the sun is between 89° and 90°. Show that the distance from the earth to the sun is at least 50 times the distance from the earth to the moon.

Solution: From Figure 5-15d we see that if the moon is exactly at half-phase the angle SME is a right angle. Since angle SEM $= \beta$ and $89^\circ \le \beta \le 90^\circ$, we have $0^\circ \le \alpha \le 1^\circ$.

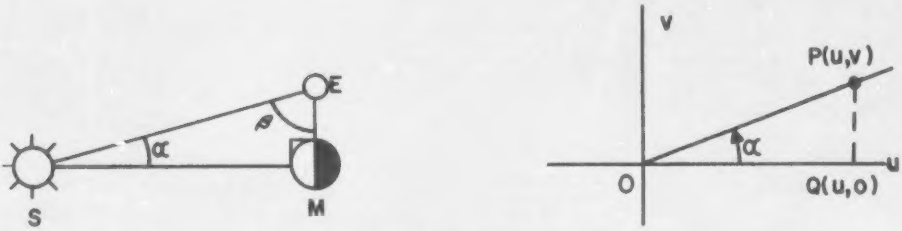


Figure 5-15d.

Let P(u,v) be a point on the terminal side of the angle α in standard position with ordinate v equal to the distance EM. Then the distance r of P from the origin will be equal to ES. By Theorem 5-15a

$$\frac{v}{r} = \sin\alpha$$

and from Table III

$$\sin\alpha \le \sin 1^\circ \approx .018 \qquad (.0175)$$

so that

$$\frac{v}{r} < .018 = \frac{18}{1000} < \frac{20}{1000} = \frac{1}{50}$$

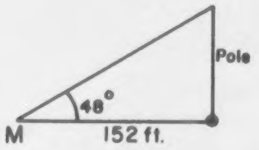
$$r > 50v.$$

Thus the distance from earth to the sun is at least 50 times the distance from earth to moon.

The essential step in these examples is the discovery and construction of a right triangle one of the sides of which is the length to be measured. Later in this section, we shall learn some additional theorems about the trigonometric functions which will permit us to use more general triangles in a similar way.

Exercises 5-15a

1. A man standing 152 feet from the foot of a flagpole, which is on his eye level, observes that the angle of elevation of the top of the flagpole is 48°. Find the height of the pole.



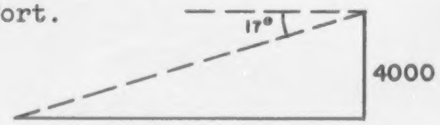
2. A wire 35 feet long is stretched from level ground to the top of a pole 25 feet high. Find the angle between the pole and the wire.

3. On a 3 per cent railroad grade, at what angle are the rails inclined to the horizontal and how far does one rise in traveling 9000 feet upgrade measured along the rails? (A grade of 3 per cent means that the tracks rise 3 feet in each 100 feet of horizontal distance.)

4. Find the radius of a regular decagon each side of which is 8 inches.

5. How long is the chord subtended by a central angle of 52° in a circle of radius 15 inches?

6. From a mountain top 4000 feet above a fort the angle of depression of the fort is 17°. Find the airline distance from the mountain top to the fort.



The law of cosines and the law of sines. In the earlier part of the section you learned how to apply the tables of sines and cosines to right triangles. We shall now develop some formulas that will enable us to study general triangles. Let ABC be a general triangle. The measure of the angle at vertex A is α, at B is β, and at C is γ. We choose α, β, and γ to be positive. Also we set $a = BC$, $b = AC$ and $c = AB$. (See Figure 5-15e.)

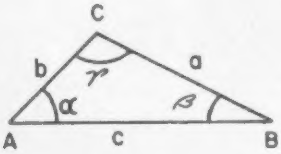


Figure 5-15e.

There are two basic relations among the parts of triangle ABC which we formulate as theorems. We begin with the law of cosines.

Theorem 5-15b. (The Law of Cosines). In triangle ABC we have

$$c^2 = a^2 + b^2 - 2ab \cos \gamma .$$

Similarly

$$b^2 = a^2 + c^2 - 2ac \cos \beta ,$$

$$a^2 = b^2 + c^2 - 2bc \cos \alpha .$$

Proof: We place triangle ABC on a rectangular coordinate system in such a way that γ is in standard position. (See Figure 5-15f.)

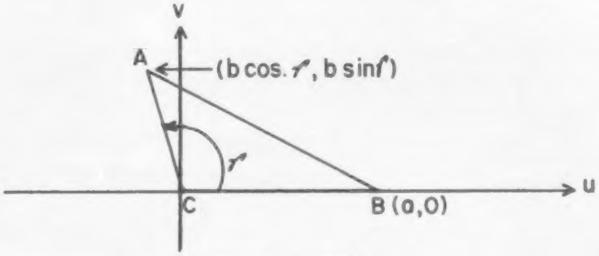


Figure 5-15f.

Then the coordinates of B are $(a,0)$ since $BC = a$. Also $AC = b$ and therefore from Theorem 5-15a we see that the coordinates of A are $(b\cos\gamma, b\sin\gamma)$. Using the distance formula we have

$$\begin{aligned} c^2 = (AB)^2 &= (b\cos\gamma - a)^2 + (b\sin\gamma - 0)^2 \\ &= b^2[(\cos\gamma)^2 + (\sin\gamma)^2] + a^2 - 2ab\cos\gamma . \end{aligned}$$

Now by definition the point $(\cos\gamma, \sin\gamma)$ lies on the unit circle, and therefore $(\cos\gamma)^2 + (\sin\gamma)^2 = 1$. Hence

$$c^2 = a^2 + b^2 - 2ab\cos\gamma .$$

This is the first formula in Theorem 5-15b. The other two formulas are obtained in a similar manner.

Example 4. In triangle ABC, $a = 10$, $b = 7$, $\gamma = 32^\circ$. Find c.

Solution: By the law of cosines

$$c^2 = 100 + 49 - 140\cos 32^\circ .$$

Using Table III

$$\cos 32^\circ \approx .848$$

and therefore $$c^2 \approx 149 - 140(.848)$$

$$\approx 30$$

Hence $$c \approx 5.5$$

Example 5. In triangle ABC, $a = 10$, $b = 7$, $c = 12$. Find α.

Solution: By the law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha .$$

Hence $$\cos\alpha = \frac{49 + 144 - 100}{2\cdot 7\cdot 12} = \frac{93}{168} \approx .554 .$$

Thus $\alpha \approx 56^\circ$ to the nearest degree.

Suppose triangle ABC is a right triangle with right angle at C, i.e., $\gamma = 90^\circ$. In this case c is the hypotenuse of the right triangle and since $\cos 90^\circ = 0$, the law of cosines becomes $c^2 = a^2 + b^2$. But this is just the Pythagorean Theorem. Therefore the law of cosines can be viewed as the generalization of the Pythagorean Theorem to arbitrary triangles. However, we do not have a new proof of the Pythagorean Theorem here, because our proof of the law of cosines depends on the distance formula which was established on the basis of the Pythagorean Theorem!

It is worth noting, though, that the law of cosines can be used to prove the converse of the Pythagorean Theorem. If, in triangle ABC we know that $c^2 = a^2 + b^2$, then we must show that $\gamma = 90^\circ$. By the law of cosines $c^2 = a^2 + b^2 - 2ab \cos \gamma$ and, combining this with $c^2 = a^2 + b^2$, we obtain $\cos \gamma = 0$. We know that $0 < \gamma < 180^\circ$, and the only angle in this range whose cosine is zero is 90°. Therefore $\gamma = 90^\circ$ as was to be proved.

Exercises 5-15b

1. Use the law of cosines in the solution of the following triangles.

   a) $\alpha = 60^\circ$, $b = 10.0$, $c = 3.0$, find a.

   b) $a = 2\sqrt{61}$, $b = 8$, $c = 10$, find α.

   c) $a = 4.0$, $b = 20.0$, $c = 18.0$, find α, β, and γ.

2. Find the largest angle of a triangle having sides 6, 8, and 12.

3. A parallelogram has two adjacent sides a, b and included angle θ.

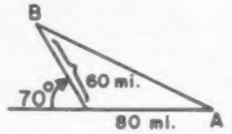
   a) Find the length of the diagonal opposite the $\angle \theta$.

   b) Find the length of the diagonal cutting the $\angle \theta$.

   c) Find the area of the parallelogram.

4. Find the sides of a parallelogram if the lengths of its diagonals are 12 inches and 16 inches and one angle formed by the diagonals is 37°.

5. A car leaves town A on a straight road and travels for 80 miles before coming to a curve in the road. At this point the direction of the road changes through an angle of 70°. The car travels 60 miles more before coming to town B. What is the straight line distance from town A to town B?



6. What interpretation can one give to the formula

$$c^2 = a^2 + b^2 - 2ab\cos\gamma, \quad \text{if } \gamma = 180^\circ?$$

7. Prove that in triangle ABC

$$\frac{1+\cos\alpha}{2} = \frac{(b+c+a)(b+c-a)}{4bc};$$

$$\frac{1-\cos\alpha}{2} = \frac{(a+b-c)(a-b+c)}{4bc}.$$

The second formula that we shall introduce relating the parts of a triangle is known as the law of sines.

Theorem 5-15c. (The Law of Sines). In triangle ABC we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

Proof: We again place triangle ABC on a rectangular coordinate system so that γ is in standard position. (See Figure 5-15g.) The coordinates of A and B are the same as in the proof of Theorem 5-15b. We know from geometry that the area of triangle ABC is $\frac{1}{2}ha$, where h is the length of the altitude from vertex A to side BC. (See Figure 5-15g.) But h is just the ordinate of A and hence $h = b \sin \gamma$. Therefore we have that the

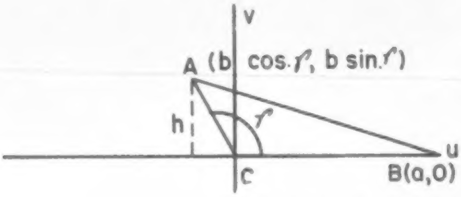


Figure 5-15g.

$$\text{area of triangle } ABC = \tfrac{1}{2}ab \sin \gamma. \tag{1}$$

Similarly we obtain

$$\text{area of triangle } ABC = \tfrac{1}{2}ac \sin \beta , \tag{2}$$

$$\text{area of triangle } ABC = \tfrac{1}{2}bc \sin \alpha . \tag{3}$$

Thus,

$$\tfrac{1}{2}bc \sin \alpha = \tfrac{1}{2}ac \sin \beta = \tfrac{1}{2}ab \sin \gamma .$$

If we divide this relation by $\frac{1}{2}abc$ we find

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

which is the desired formula.

Example 6. If, in triangle ABC, $a = 10$, $\beta = 42^\circ$, $\gamma = 51^\circ$, find b.

Solution: Since $\alpha + \beta + \gamma = 180^\circ$ we have $\alpha = 87^\circ$. By the law of sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b},$$

$$b = \frac{a \sin\beta}{\sin\alpha} = \frac{10 \sin 42^\circ}{\sin 87^\circ} \approx \frac{6.69}{.999} \approx 6.7.$$

Example 7. Find the area of triangle ABC if $a = 10$, $b = 7$, $\gamma = 68^\circ$.

Solution: According to (1), the area of triangle ABC $= \frac{1}{2}ab \sin\gamma = 35 \sin 60^\circ \approx 35(.927) \approx 32.4$.

Example 8. Are there any triangles ABC such that $a = 10$, $b = 5$, and $\alpha = 22^\circ$?

Solution: Before attempting to solve Example 8, let us try to construct a triangle ABC geometrically, given a, b, and α. At point A on a horizontal line construct angle α. Lay off side AC of length b on the terminal side of angle α. Now with C as center strike an arc of radius a. There are a number of possibilities depending on a, b, and α which are illustrated in Figure 5-15h.

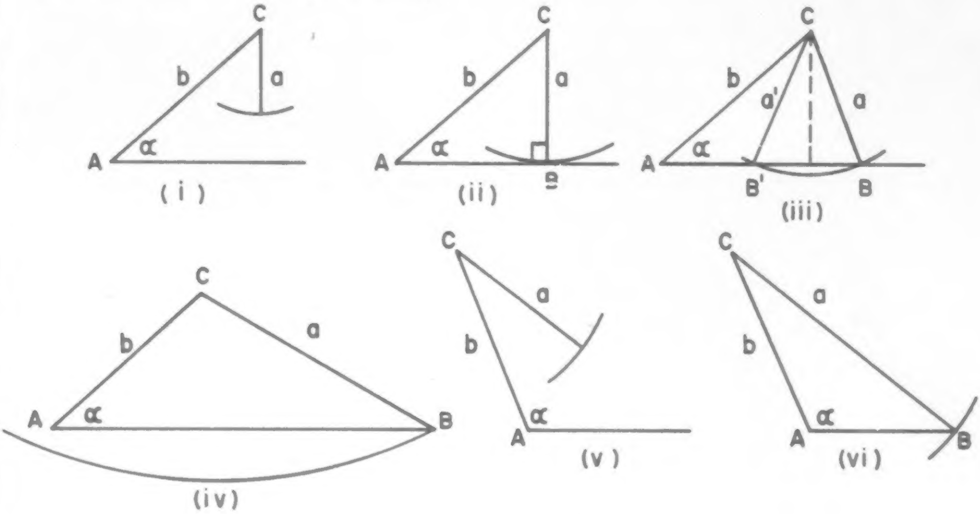


Figure 5-15h.

In case (i) there is no triangle;
in case (ii) there is one triangle;
in case (iii) there are two triangles;
in case (iv) there is one triangle;
in case (v) there is no triangle;
in case (vi) there is one triangle.

Thus to solve Example 8, we attempt to find β keeping in mind that there may be zero, one, or two solutions. If such a triangle exists, then by the law of sines

$$\frac{\sin\beta}{5} = \frac{\sin 22^\circ}{10}$$

or
$$\sin\beta = \frac{1}{2}\sin 22^\circ \approx .187.$$

Recall that $\sin\beta$ is positive if $90^\circ < \beta < 180^\circ$, and if $\beta = 180^\circ - \theta$ where $0 < \theta < 90^\circ$ then $\sin\beta = \sin\theta$. Thus from $\sin\beta \approx .187$ we conclude that $\beta = 11^\circ$ or $\beta = 169^\circ$ to the nearest degree. Are both of these values of β possible? If $\beta = 169^\circ$, then $\alpha + \beta = 191^\circ$ which is impossible. Why? Therefore, there is one triangle with the given data. We are in case (iv).

Example 9. Are there any triangles ABC with $a = 15$, $b = 10$, and $\alpha = 105^\circ$?

Solution: We attempt to find β. If there is such a triangle we have, from the law of sines,

$$\frac{\sin\beta}{10} = \frac{\sin\alpha}{15}.$$

But $\sin\alpha = \sin 105^\circ = \sin(180^\circ - 75^\circ) = \sin 75^\circ \approx .966$. Hence, $\sin\beta \approx \frac{2}{3}(.966) = .644$ and this implies $\beta = 40^\circ$ or $\beta = 140^\circ$. Clearly β can not be 140° and there is one triangle with the given data. This is an example of case (vi).

Example 10. Are there any triangles ABC such that $a = 10$, $b = 50$, and $\alpha = 22^\circ$?

Solution: We attempt to find β. If there is such a triangle we have, from the law of sines,

$$\frac{\sin\beta}{50} = \frac{\sin 22^\circ}{10},$$

$$\sin\beta = 5(.375) > 1.$$

But we know that the sin never exceeds one, and therefore our assumption that a triangle with the given data exists leads to a contradiction. Thus there are no such triangles. This is an illustration of case (i).

Exercises 5-15c

1. Use the law of sines in the solution of the following:

   a) $\beta = 68^\circ$, $\gamma = 30^\circ$, $c = 22.0$, find a.

   b) $\alpha = 45^\circ$, $\gamma = 60^\circ$, $b = 20.0$, find a and c.

   c) $\gamma = 30^\circ$, $b = 5$, $c = 2$, find α.

   d) $c = 620$, $b = 480$, $\gamma = 55^\circ$, find α and a.

   e) $\gamma = 65^\circ$, $b = 97$, $c = 91$, find β.

   f) $a = 80$, $b = 100$, $\alpha = 36^\circ$, find β.

   g) $a = 31$, $b = 50$, $\alpha = 33^\circ$, find β.

   h) $a = 50$, $b = 60$, $\gamma = 111^\circ$, find α and c.

2. Prove that in triangle ABC

$$\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{a - b}{a + b}.$$

5-16. Trigonometric Identities and Equations.

The following exercises can be used to provide additional practice in trigonometric equations and identities.

Exercises 5-16

Prove the following identities:

1. $\tan x \cos x = \sin x$
2. $(1 - \cos x)(1 + \cos x) = \sin^2 x$
3. $\frac{\cos x}{1 + \sin x} = \frac{1 - \sin x}{\cos x}$.
4. $\tan x = \frac{\sin 2x}{1 + \cos 2x}$
5. $\tan x \sin 2x = 2 \sin^2 x$
6. $1 - 2 \sin^2 x + \sin^4 x = \cos^4 x$
7. $\frac{2 \cos^2 x - \sin^2 x + 1}{\cos x} = 3 \cos x$
8. $\sin x \tan x + \cos x = \frac{1}{\cos x}$
9. $\frac{1}{\cos^2 x} + \tan^2 x + 1 = \frac{2}{\cos^2 x}$
10. $\sin^4 x - \sin^2 x \cos^2 x - 2 \cos^4 x = \sin^2 x - 2 \cos^2 x$
11. $\frac{\cos^4 x - \sin^4 x}{1 - \tan^4 x} = \cos^4 x$
12. $\sin 4x = 4 \sin x \cos x \cos 2x$
13. $\cos^4 x - \sin^4 x = 1 - 2 \sin^2 x$
14. $\frac{\sin (x + y)}{\cos x \cos y} = \tan x + \tan y$
15. $\sin (x + y) + \sin (x - y) = 2 \sin x \cos y$
16. $\cos^4 x - \sin^4 x = 2 \cos^2 x - 1$
17. $\cos (x + y) - \cos (x - y) = -2 \sin x \sin y$
18. $\sin (x + y) - \sin (x - y) = 2 \cos x \sin y$
19. $\cos (x + y) + \cos (x - y) = 2 \cos x \cos y$

20. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

21. $\frac{\sin 2x}{1+\cos 2x} = \frac{1-\cos 2x}{\sin 2x}$

22. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

23. $\frac{\sin x}{1+\cos x} = \tan\frac{x}{2}$

24. $3\sin x - \sin 3x = 4\sin^3 x$

25. Prove by counter-example that none of the following is an identity:

a) $\cos(x-y) = \cos x - \cos y$

b) $\cos(x+y) = \cos x + \cos y$

c) $\sin(x-y) = \sin x - \sin y$

d) $\sin(x+y) = \sin x + \sin y$

e) $\cos 2x = 2\cos x$

f) $\sin 2x = 2\sin x$

Solve each of the following equations:

26. $2\sin x - 1 = 0$

27. $4\cos^2 x - 3 = 0$

28. $3\tan^2 x - 1 = 0$

29. $\sin^2 x - \cos^2 x + 1 = 0$

30. $2\cos^2 x - \sqrt{3}\cos x = 0$

31. $4\sin^3 x - \sin x = 0$

32. $2\sin^2 x - 5\sin x + 2 = 0$

33. $2\sin x\cos x + \sin x = 0$

34. $\cos x + \sin x = 0$

35. $2\sin^2 x + 3\cos x - 3 = 0$

36. $\cos 2x = 0$

37. $\cos 2x - \sin x = 0$

38. $2\cos^2 x + 2\cos 2x = 1$

39. $\cos 2x + 2\cos^2(\frac{x}{2}) = 1$

40. $6\sin^2 x + \cos 2x = 2$

41. $\sin 2x - \cos^2 x + \sin^2 x = 0$

42. $2\cos^2 x + \cos 2x = 2$

43. $\cos 2x - \cos x = 0$

44. $2\sin^2 x - 3\cos x - 3 = 0$

45. $\cos 2x \cos x + \sin 2x \sin x = 1$

46. $\cos^2 x - \sin^2 x = \sin x$

47. $\cos x = \frac{1 + \cos^2 x}{2}$

48. $3\sin x + \sqrt{3}\cos x = 0$

49. $\tan x = x$

50. $\pi \sin x = 2x$

5-17. Calculation of sin x and cos x. Euler's Formula.

We wish to find polynomials whose values approximate $\sin x$ and $\cos x$ for x near zero. The process is analogous to that used in Section 4-16 to approximate e^x, except that in the present case the required polynomials are developed simultaneously.

Let us use f to denote sin and g to denote cos, so that $f: x \longrightarrow \sin x$ and $g: x \longrightarrow \cos x$. Let us begin with $g: x \longrightarrow \cos x$. Since g is an even function (that is, $\cos(-x) = \cos x$), we seek approximations which include only even powers. Thus we seek approximations of the form

$$\cos x \approx b_0 \qquad (0)$$

$$\cos x \approx b_0 + b_2x^2 \qquad (2)$$

$$\cos x \approx b_0 + b_2x^2 + b_4x^4 \qquad (4)$$

$$\cdots$$

We know from Section 5-10 that the best linear approximation to $\cos x$ is $1 + 0 \cdot x = 1$. We therefore choose $b_0 = 1$. We wish to determine the proper coefficients $b_2, b_4, \dots$.

For the function $f: x \longrightarrow \sin x$, $f(-x) = -f(x)$. It is therefore necessary to use only odd powers in the approximating polynomials. Using the fact that the best linear approximation is x (see Section 5-10), we seek approximations of the following form:

$$\sin x \approx x \qquad (1)$$

$$\sin x \approx x + a_3 x^3 \qquad (3)$$

$$\sin x \approx x + a_3 x^3 + a_5 x^5 \qquad (5)$$

where the coefficients $a_3, a_5, \dots$ remain to be determined.

Now $$\cos' x = -\sin x. \qquad (6)$$

We wish to replace $\cos x$ and $\sin x$ by polynomial approximations. We do this in such a way that the slope function of a $\cos x$ approximation will be the negative of a polynomial approximation for $\sin x$. This is possible only if the degree of the $\sin x$ approximation is one less than the $\cos x$ approximation. Thus, from (6),

$$\cos x \approx 1 + b_2 x^2, \quad \text{and} \quad \sin x \approx x,$$

we obtain

$$2b_2 x = -x$$

so that $$b_2 = -\frac{1}{2}$$

and $$\cos x \approx 1 - \frac{1}{2}x^2. \qquad (2')$$

Similarly, in $\sin' x = \cos x$, we replace $\sin x$ by a polynomial approximation and $\cos x$ by the approximation of degree one less.

Thus using

$$\sin x \approx x + a_3x^3 \tag{3}$$

and

$$\cos x \approx 1 - \frac{x^2}{2}, \tag{2'}$$

we have

$$1 + 3a_3x^2 = 1 - \frac{x^2}{2},$$

so that $a_3 = -\frac{1}{2\cdot 3}$

and $\sin x \approx x - \frac{x^3}{3!}$.

Continuing in this way we obtain the following approximations for sin x and cos x.

$\sin x$	$\cos x$
x	1
$x - \frac{x^3}{3!}$	$1 - \frac{x^2}{2!}$
$x - \frac{x^3}{3!} + \frac{x^5}{5!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$
	...

For example, $\sin 0.2 \approx .2 - \frac{.008}{6} + \frac{.00032}{120}$

$$\approx .19867$$

and $\cos 0.2 \approx 1 - \frac{.04}{2} + \frac{.0016}{24}$

$$\approx .98007$$

(The values given in Table I are sin 0.2 = .1987, cos 0.2 = .9801)

We shall not prove that the polynomials written represent the best approximations possible for the degree chosen. (They do.) Nor shall we discuss the greatest possible error made in using one of these approximations. We assert without proof that $\sin x$ is between any two of the successive polynomial approximations listed, in particular, between

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

and

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Similarly, $\cos x$ is between

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

and

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

There is a remarkable relation, due to Euler, between the exponential function on the one hand and the circular functions, cos and sin, on the other. This relation is expressed by the famous equation

$$e^{ix} = \cos x + i \sin x. \qquad (7)$$

The symbol e^{ix}, with an imaginary exponent, obviously requires an interpretation.

To interpret (7) geometrically, we draw a unit circle in the complex plane (see Figure 5-17) with x the arc length measured

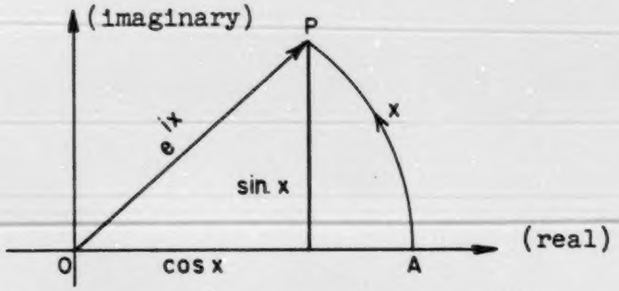


Figure 5-17. Interpretation of e^{ix}.

counter-clockwise from $A(1 + 0i)$ to the point P. The vector $\overrightarrow{OP}$ from the origin to P represents $\cos x + i \sin x$, and therefore, according to equation (7), it represents e^{ix} also. This is entirely sensible. For, if

$$e^{ix} = \cos x + i \sin x$$

and

$$e^{iy} = \cos y + i \sin y,$$

and if we assume that imaginary exponents are added like real ones, we obtain by multiplication

$$e^{ix + iy} = (\cos x + i \sin x)(\cos y + i \sin y),$$

and hence

$$e^{i(x + y)} = (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y). \qquad (8)$$

By (7)

$$e^{i(x + y)} = \cos(x + y) + i \sin(x + y). \qquad (9)$$

From Equations (8) and (9) we immediately obtain the addition formulas for $\cos(x + y)$ and $\sin(x + y)$.

Let us return to Equation (7) and consider the approximations that we have made. The fourth degree approximation to e^x is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}. \qquad (10)$$

If in (10) we replace x by ix, we obtain

$$e^{ix} \approx 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!}.$$

That is,

$$\cos x + i \sin x \approx 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!}$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) + i\left(x - \frac{x^3}{3!}\right).$$

This means that real and imaginary parts are approximately equal so that

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

and

$$\sin x \approx x - \frac{x^3}{3!}.$$

These results agree with the approximations previously obtained.

Exercises 5-17

1. Compute $\sin 0.1$ correct to 4 places of decimals, and compare with Table I.

2. Over what x interval can we use the approximations

   a) $\sin x \approx x$

   b) $\cos x \approx 1$

   if the error is to be less than 0.01?

3. Use the identity, $\tan x = \frac{\sin x}{\cos x}$, and the polynomial approximations to $\sin x$ and $\cos x$ of fifth and fourth degree respectively, to obtain by long division a polynomial approximation to $\tan x$. What does this approximation suggest as a relationship between $\tan(-x)$ and $\tan x$?

4. Use Euler's equation to find e^{ix} if

   a) $x = \frac{\pi}{2}$

   b) $x = \pi$

   c) $x = 3\pi$

   d) $x = \frac{\pi}{3}$

   e) $x = 0.5$

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